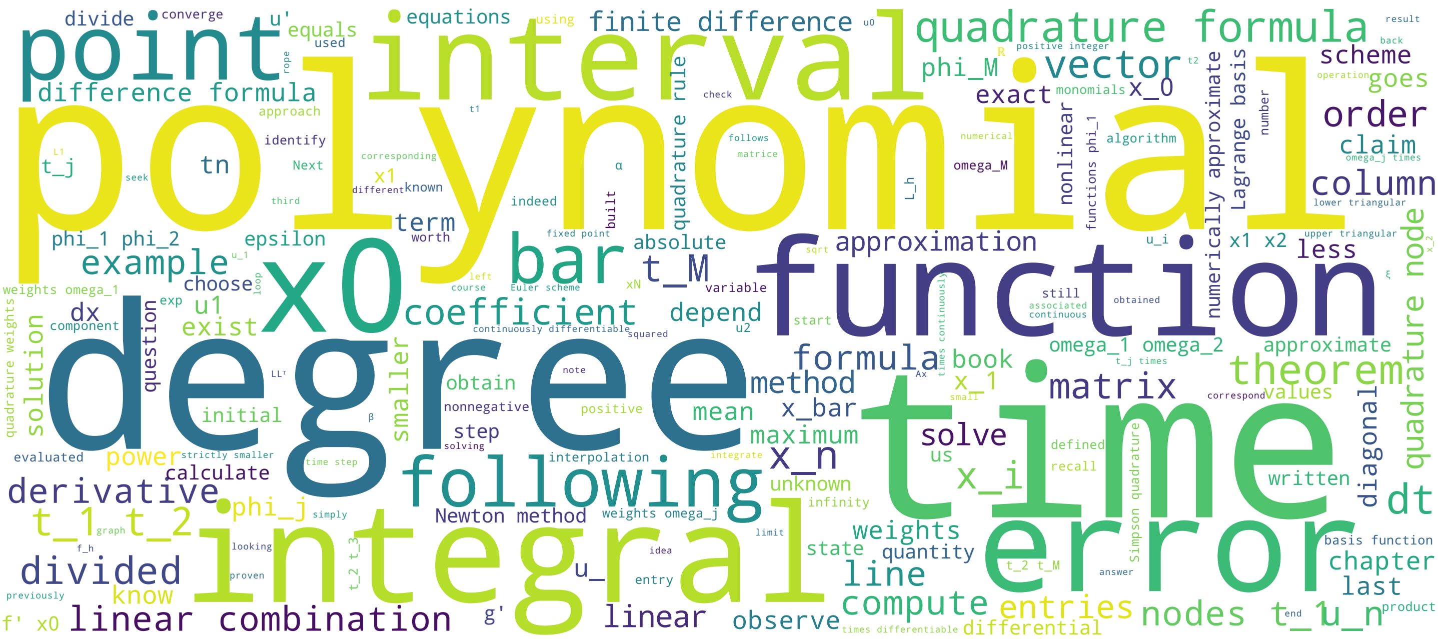
Chap. 3 : poids d'une formule de quadrature



Video

Chap.3 : poids d'une formule de quadrature

- Soit $J(g) = \sum_{j=1}^{M} w_j g(t_j)$ une formule de quadrature pour approcher num. $\int_{-1}^{1} g(t) dt$
- $t_1, t_2, \ldots, t_M$ donnés, calcul de $w_1 w_2 \ldots w_M$?
- Idée : Soit $p \in \mathbb{P}_{M-1}$. D'après le chap.1 on a $p(t) =$ $\varphi_1(t)$ $\varphi_2(t)$ $\varphi_M(t)$

  $\varphi_1 \varphi_2 \ldots \varphi_M$ est la base de Lagrange de $\mathbb{P}_{M-1}$ associée aux pts $t_1 t_2 \ldots t_M$.

Let us now consider the calculation of the quadrature weights. Let J(g) be the sum over from 1 to M of omega_j times g(t_j) be a quadrature formula. such a quadrature formula is used to numerically approximate the integral between -1 and 1 of g(t) dt. Choosing a quadrature formula comes down to specifying integration nodes t_1, t_2,...,t_M, and quadrature weights omega_1, omega_2,...,omega_M. By doing the linear combination of these values I seek to approximate the integral between -1 and 1 of g(t) dt. The question is now the following: given the nodes t_1, t_2 up to t_M, how to calculate the weights omega_1, omega_2 up to omega_M? The idea is the following. Let p a polynomial of degree lower or equal to m-1. From chapter one: interpolation, I claim that p(t) can be written as a linear combination of the functions phi_1(t), phi_2(t) up to phi_M(t) where phi_1, phi_2 and so on up to phi_M is the Lagrange basis of polynomials of degree smaller or equal to M-1 associated to the quadrature nodes t_1, t_2 up to t_M. jusqu'à tm. Thus p(t) is a linear combination of the functions phi_1, phi_2 and do on up to phi_M with phi_1, phi_2 up to phi_M the Lagrange basis associated to the nodes t_1, t_2 up to t_M.

Notes

Summary

0m 00s

Chap.3 : poids d'une formule de quadrature

- Soit $J(g) = \sum_{j=1}^{M} \omega_j\, g(t_j)$ une formule de quadrature pour approcher num. $\int_{-1}^{1} g(t)\,dt$
- $t_1, t_2, \dots, t_M$ données, calcul de $\omega_1 \omega_2 \dots \omega_M$?
- Idée : Soit $p \in \mathbb{P}_{M-1}$. D'après le chap.1 on a $p(t) = p(t_1)\,\varphi_1(t) + p(t_2)\,\varphi_2(t) + \dots + p(t_M)\,\varphi_M(t)$

$\varphi_1\, \varphi_2 \dots \varphi_M$ est la base de Lagrange de $\mathbb{P}_{M-1}$ associée aux pts $t_1\, t_2 \dots t_M$.

$$\varphi_1(t) = \frac{(t-t_2)(t-t_3)\cdots(t-t_M)}{(t_1-t_2)(t_1-t_3)\cdots(t_1-t_M)}$$

Donc

$$\int_{-1}^{1} p(t)\,dt = \sum_{j=1}^{M} p(t_j) \underbrace{\int_{-1}^{1} \varphi_j(t)\,dt}_{\omega_j}.$$

Donc si $\omega_j = \int_{-1}^{1} \varphi_j(t)\,dt$ $\int_{-1}^{1} p(t)\,dt = J(p)$ $\forall \mathbb{P}_{M}$

I recall, for example that phi_1(t) was the polynomial of degree M-1 which is worth 0 in the nodes t_2, t_3 up to t_M, and is worth 1 in t_1. Hence I divide by (t_1-t_2) (t_1-t_3)...(t_1-t_M), this polynomial phi_1 (t) is indeed a polynomial of degree M-1, it is the product of monomials, M-1 monomials in total and it is a polynomial equal to 0 in the points t_2, t_3.. t_M and equal to 1 in t_1. Sine we have proven that these functions form a basis of polynomials of degree M-1, thus I can write any polynomial of degree M-1 as a linear combination of these basis functions. Moreover, the coefficients of the linear combination are given by p(t_1), p(t_2) up to p(t_M). de p de t. Now if we use this representation formula of p(t) and integrate between -1 and 1 on both sides we obtain the following: the integral between -1 and 1 of p(t) dt is equal to the sum over j=1 to M of p(t_j) times the integral between -1 and 1 of phi_j(t) dt. Let us denote omega_j this quantity, I have obtained a quadrature formula and this formula is exact for polynomials of degree M-1. If omega_j is equal to the integral over -1 and 1 of phi_j(t) dt, I have built a quadrature formula such that the integral over -1 to 1 of p(t) is equal to J(p) for all polynomial p of degree M-1.

Notes

Summary

2m 37s

Chap. 3 : poids d'une formule de quadrature

- Soit $J(g) = \sum_{j=1}^{M} \omega_j g(t_j)$ une formule de quadrature pour approcher num. $\int_{-1}^{1} g(t)dt$
- $t_1, t_2, \dots, t_M$ donnés, calcul de $\omega_1 \omega_2 \dots \omega_M$?
- Idée : Soit $p \in \mathbb{P}_{M-1}$. D'après le chap. 1 on a $p(t) = p(t_1)\varphi_1(t) + p(t_2)\varphi_2(t) + \dots + p(t_M)\varphi_M(t)$

  $\varphi_1 \varphi_2 \dots \varphi_M$ est la base de Lagrange de $\mathbb{P}_{M-1}$ associée aux pts $t_1 t_2 \dots t_M$.

$$\varphi_1(t) = \frac{(t-t_2)(t-t_3)\cdots(t-t_M)}{(t_1-t_2)(t_1-t_3)\cdots(t_1-t_M)}$$

Donc
$$\int_{-1}^{1} p(t)dt = \sum_{j=1}^{M} p(t_j) \underbrace{\int_{-1}^{1} \varphi_j(t)dt}_{\omega_j}.$$
Donc si $\omega_j = \int_{-1}^{1} \varphi_j(t)dt$ $\int_{-1}^{1} p(t)dt = J(p)$ $\forall \mathbb{P}_{M-1}$

- Thm 3.2 : Donnée : M, $t_1, t_2, \dots t_M$, $\omega_1 \omega_2 \dots \omega_M$ $J(g) = \sum_{j=1}^{M} g(t_j)\omega_j$ pour appr. $\int_{-1}^{1} g(t)dt$

$$\left(\int_{-1}^{1} p(t)dt = J(p) \quad \forall p \in \mathbb{P}_{M-1} \right) \iff \left(\omega_j = \int_{-1}^{1} \varphi_j(t)dt \quad j = 1, \dots, M \right)$$

I have therefore built a quadrature formula which is excat for polynomials of degree M-1. I can now state a theorem which allows us to compute the weights given the nodes of a quadrature formula. Theorem 3.2 of the book. Let J(.) be a quadrature formula that is a positive integer M, t_1, t_2 up to t_M some quadrature nodes, omega_1, omega_2 up to omega_M the weights, thus J(g) is equal to the sum over j from 1 to M of g(t_j) times omega_j, J(g) being here to numerically approximate the integral over -1 and 1 of g(t) dt. I state the following: the quadrature formula is exact for all polynomials of degree M-1, that is ti say the integral between -1 and 1 of p(t) dt, Ihere p is any polynomial of degree M-1, the integral over -1 to 1 of p(t)dt is equal to J(p) for all p, polynomial of degree M-1 if and only if the weights omega_j are equal to the integral between -1 and 1 of phi_j(t) dt, for all j from 1 to M. Here the functions phi_j(t) are the basis function of P_M associated to the nodes t_1, t_2,...,t_M. We now have a formula for the weights. Donc, qu'est-ce qu'on va faire maintenant ? The interpretation of theorem 3.2 is the following.

Notes

Summary

5m 21s

Chap.3 : poids d'une formule de quadrature

- Soit $J(g) = \sum_{j=1}^{M} w_j\, g(t_j)$ une formule de quadrature pour approcher num. $\int_{-1}^{1} g(t)\,dt$
- $t_1, t_2, \dots, t_M$ donnés, calcul de $w_1 w_2 \dots w_M$?
- Idée : Soit $p \in \mathbb{P}_{M-1}$. D'après le chap.1 on a $p(t) = p(t_1)\,\varphi_1(t) + p(t_2)\,\varphi_2(t) + \dots + p(t_M)\,\varphi_M(t)$

  $\varphi_1\, \varphi_2 \dots \varphi_M$ est la base de Lagrange de $\mathbb{P}_{M-1}$ associée aux pts $t_1\, t_2 \dots t_M$.

$$\varphi_1(t) = \frac{(t-t_2)(t-t_3)\cdots(t-t_M)}{(t_1-t_2)(t_1-t_3)\cdots(t_1-t_M)}$$

Donc

$$\int_{-1}^{1} p(t)\,dt = \sum_{j=1}^{M} p(t_j) \underbrace{\int_{-1}^{1} \varphi_j(t)\,dt}_{w_j}.$$

Donc si $w_j = \int_{-1}^{1} \varphi_j(t)\,dt$ $\quad \int_{-1}^{1} p(t)\,dt = J(p)$ $\forall \mathbb{P}_{M-1}$

- Thm 3.2 : Donnée : M, $t_1, t_2, \dots t_M$, $w_1 w_2 \dots w_M$ $\quad J(g) = \sum_{j=1}^{M} g(t_j) w_j$ pour appr. $\int_{-1}^{1} g(t)\,dt$

$$\left(\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_{M-1} \right) \iff \left(w_j = \int_{-1}^{1} \varphi_j(t)\,dt \quad j = 1, \dots, M \right)$$

We choose quadrature nodes t_1, t_2 up to t_M in the interval -1 to 1, We compute the weights omega_j: integratal from -1 to 1 of phi_j(t) dt for j from 1 to M. Let us go back to theorem 3.1. I claim that the integral between a and b of f(x) dx minus the corresponding approximation L_h(f), well this error is of order h to the power M, iprovided f is M times differentiable in the interval [a,b]. This means that the error is divided by 2 to the power M each time h is divided by two. Now we will build, thanks to this formula of the weights the Simpson quadrature rule using three points. Then we shall ask ourselves: Does there exist a good choice of the quadrature nodes t_1, t_2 up to t_M? First, the Simpson quadrature rule.

Notes

Summary

7m 22s