Dérivées numériques d'ordre 2

Introduction à l'analyse numérique

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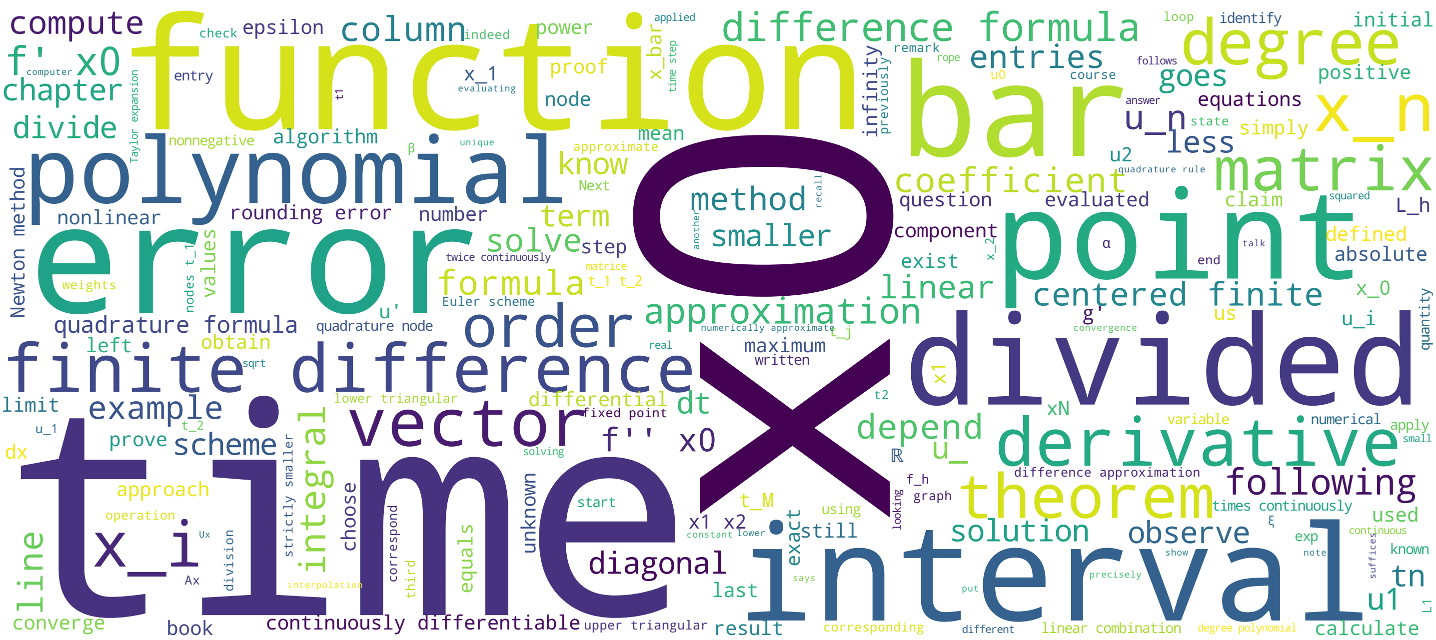

Video

Dériv. num. ordre 2

$f: \mathbb{R} \to \mathbb{R} \quad C^2 \quad x_0 \in \mathbb{R} \qquad f''(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h/2) - f'(x_0 - h/2)}{h}$

$$f'(x_0 + \tfrac{h}{2}) \simeq \frac{f(x_0 + h/2 + h/2) - f(x_0 + h/2 - h/2)}{h} \qquad f'(x_0 - h/2) \simeq \frac{f(x_0 - h/2 + h/2) - f(x_0 - h/2 - h/2)}{h}$$

$$f''(x_0) \simeq \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

$$\left| f''(x_0) - \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2} \right| = O(h^2)$$

Let us now consider second order derivatives. Let f be a real valued function, twice continuously differentiable, let x0 be a real value, We now want to approach f''(x0). If I apply one of the definitions of the derivative of f'(x0) I get the limit when h goes to zero of f'(x0+h/2) minus f'(x0-h/2) divided by h. I just applied the previous formula for f'(x0) to f''(x0). je l'applique à f seconde. Now, I approach f'(x0+h/2) by the centered finite difference formula, that is to say, f(x0+h/2+h/2) minus f(x0+h/2-h/2), and I divide by h. I do the same for f'(x-h/2), that is to say, I approach it by f(x0-h/2+h/2) minus f(x0-/h2-h/2) and I divide by h. Thus f''(x0) will be approached by, I must take the difference of these two terms, and I obtain f(x0+h) Plus h sur deux, c'est-à-dire f de x0 plus h. minus 2 f(x0) plus f(x0-h) and I must divide by h^2. We are now going to prove that this approximation of f''(x0) this centered finite difference approximation, centered since I used a centered finite difference approximation of f'(x0), this approximation is of order h^2, which is not surprising since I used centered finite difference formula.

Notes

Summary

0m 04s

Dériv. num. ordre 2

$f : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^2$ $x_0 \in \mathbb{R}$ $f''(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h/2) - f'(x_0 - h/2)}{h}$

$f'(x_0 + \frac{h}{2}) \simeq \frac{f(x_0 + h/2 + h/2) - f(x_0 + h/2 - h/2)}{h}$ $f'(x_0 - h/2) \simeq \frac{f(x_0 - h/2 + h/2) - f(x_0 - h/2 - h/2)}{h}$

$$f''(x_0) \simeq \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

$$\left| f''(x_0) - \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2} \right| = O(h^2)$$

Thm 2.5 : $\forall f \in \mathcal{C}^4$ $\forall x_0 \in \mathbb{R}$ $\exists C > 0$ $\forall 0 < h \leq 1$ $\left| f''(x_0) - \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2} \right| \leq C h^2.$

Rem : C dep de f x_0 pas de h

Interp : Choisit, f x_0 l'erreur est divisée par 4 chaque fois que h est divisé par 2.

Rem : erreur d'arrondis $O\left(\frac{1}{h^2}\right)$

Dem : $f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f^{(4)}(\xi)$ $x_0 \leq \xi \leq x_0 + h$

$f(x_0 - h) = \ldots$

More precisely, We are going to prove theorem 2.5 of the book which says: For all f four times continuously differentiable, For all x0 in R, there exists a positive C, which will therefore depend of f and x0, such that for all 0<h<1, C will not depend on h, the error between f''(x0) and its approximation with the above centered finite difference formula, again 2f(x0)-f(x0+h)-f(x0-h) divided by h^2, this error is less or equal C h^2. à h carré. As before, C depends on f and x0, but not on h. As before, the corresponding experiment is to choose f and x0, to report the error, the difference between f''(x0) and this finite difference formula, and to observe that this error is divided by four whenever h is divided by 2. One more remark about rounding errors. With this formula, rounding errors will behave like one over h^2, and no more like one over h, simply because I am evaluating second derivatives and there is a division by h^2 in the formula. Finally, the proof of the theorem is left as an exercice, but you should be able to do the proof by yourself. It suffices to take the Taylor expansion f(x0+h)=f(x0)+hf'(x0)+h^2/2f''(x0) +h^/6f'''(x0) +h^4/24f''''(xi) xi between x0 and x0+h. Do the same with f(x0-h) take the sum of these two equations and prove the result.

Notes

Summary

2m 58s