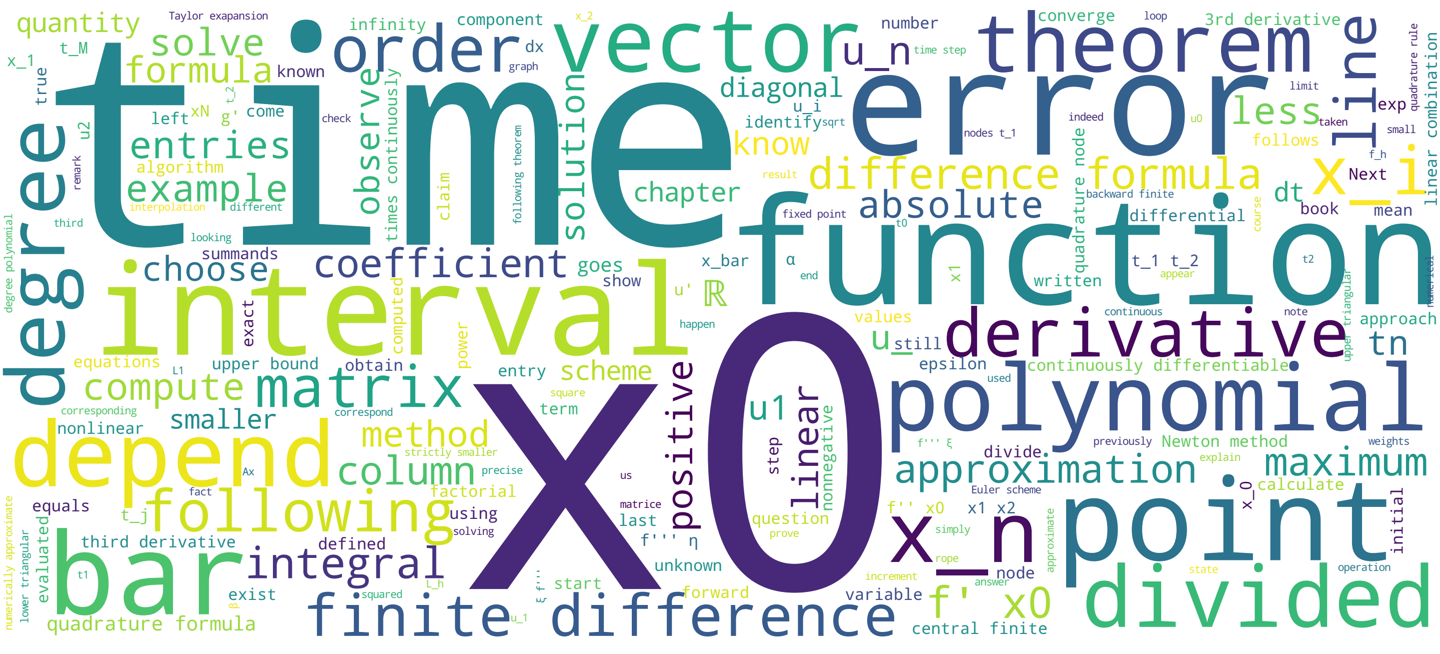
Dérivées numériques d'ordre 1 – Formule de différences finies centrée (1)

Introduction à l'analyse numérique

Prof. Marco Picasso



Video

EPFL

Dériv. Num. d'ordre 1 : FDF centrée

$$\left| f'(x_0) - \frac{f(x_0+h/2) - f(x_0-h/2)}{h} \right| = O(h^2) \ ?$$

Soit $f : \mathbb{R} \to \mathbb{R}$ C^3, soit $x_0 \in \mathbb{R}$, soit $h > 0$.

$$f(x_0+h/2) = f(x_0) + \frac{h}{2} f'(x_0) + \frac{h^2}{4.2} f''(x_0) + \frac{h^3}{8.6} f'''(\xi) \qquad x_0 \leq \xi \leq x_0 + h/2$$

$$f(x_0-h/2) = f(x_0) - \frac{h}{2} f'(x_0) + \frac{h^2}{4.2} f''(x_0) - \frac{h^3}{8.6} f'''(\eta) \qquad x_0 - h/2 \leq \eta \leq x_0$$

$$f(x_0+h/2) - f(x_0-h/2) = h f'(x_0) + \frac{h^3}{8.6} \left(f'''(\xi) + f'''(\eta) \right)$$

Let's now explain what happens for the central finite difference formula Unlike the forward and backward finite difference formula, the error between f'(x0) and the central finite difference formula isn't of order 1 but order 2 in h. Let's precise this. Let f a function from ℝ to ℝ, now three times continuously differentiable, let x0 in ℝ and let h be a positive number. Now the Taylor exapansion is the following. Take f(x0+h/2) it is f(x0) plus the increment, h/2 times f'(x0) plus the square of the increment, h²/4 divided by factorial 2, so 2 times f''(x0) plus our last term in the Taylor exapansion, the cube of the increment, h³/8, divided by factorial 3 so 2*3=6, times the third derivative at some point ξ, ξ being a point between x0 and x0+h/2. We do the same for f(x0-h/2): f(x0)-(h/2)*f'(x0)+*(h²/(4*2))*f''(x0)-... ... (h³/(8*6))*f'''(η) for some point η, η between x0-h/2 and x0. Take the difference between those two lines since that's what we have in the formula, so the difference, f(x0+h/2)-f(x0-h/2) then the summands in f(x0) cancel each other, and we're left with h*f'(x0). The summands in h² cancel each other again, that's why we get a higher order and we're left with +h³/(8x6) (f'''(ξ)+f'''(η)).

Notes

Summary

0m 05s

Dériv. Num. d'ordre 1 : FDF centrée

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Soit $f : \mathbb{R} \to \mathbb{R}$ C^3, soit $x_0 \in \mathbb{R}$, soit $h > 0$.

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$$f(x_0+h/2) - f(x_0-h/2) = h f'(x_0) + \frac{h^3}{8 \cdot 6} \left(f'''(\xi) + f'''(\eta) \right)$$

Thm 2.3 : $\forall f \in C^3 \ \forall x_0 \in \mathbb{R} \ \exists C > 0 \ \forall 0 < h \leq 1$ $\left| f'(x_0) - \frac{f(x_0+h/2) - f(x_0-h/2)}{h} \right| \leq C h^2.$

Rem : C dép de f, x_0 mais de h

Interpr : Choisit f, x_0 l'erreur est divisée par $2^2 = 4$ ($10^2 = 100$) chaque fois que h est divisée par 2 (10)

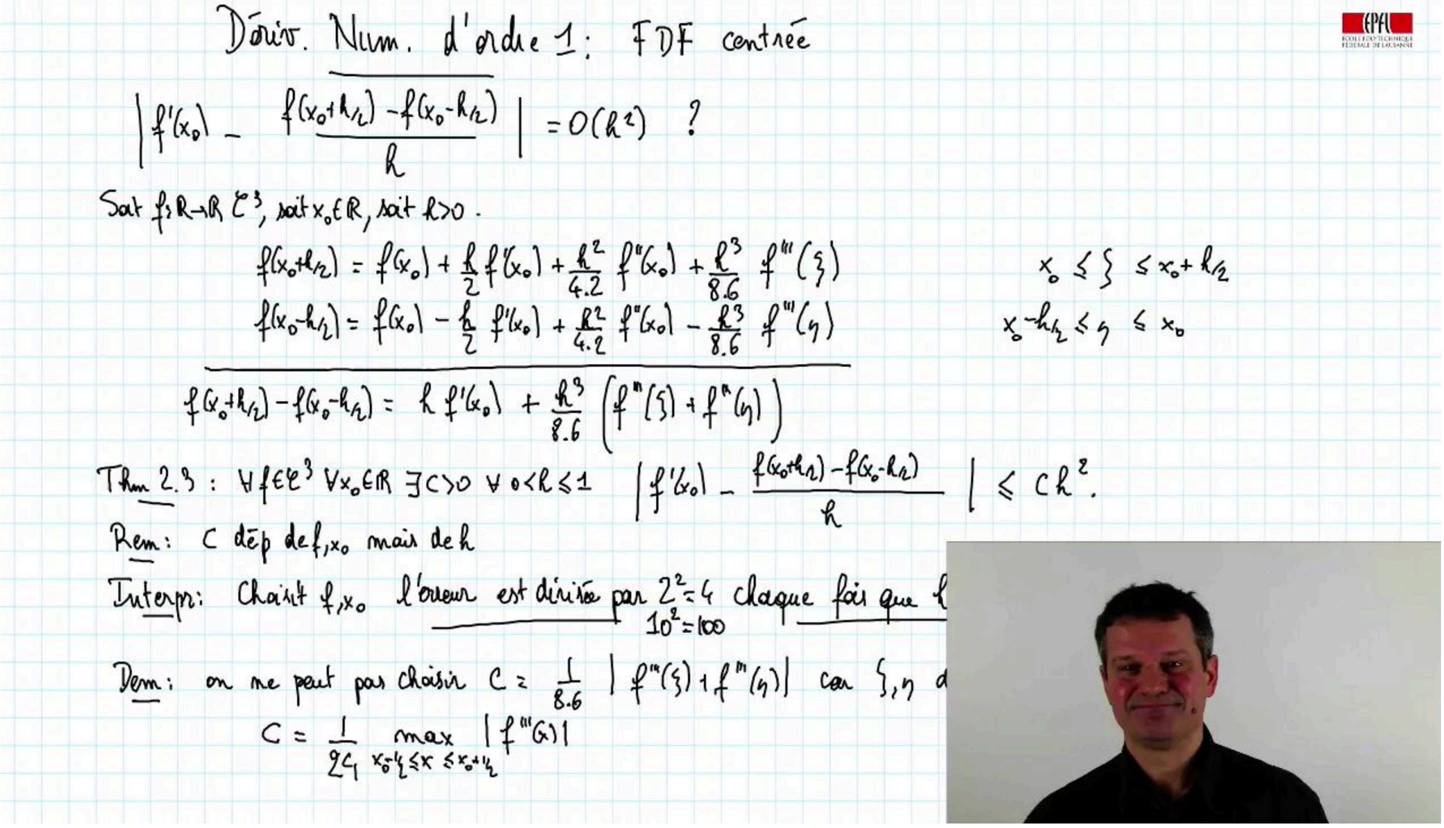
Dem :

Now I claim that the following theorem is true, it is the Theorem 2.3 in the book. For all function f 3 times continuously differentiable, since we went until the 3rd derivatives, for all x0 in ℝ there is a positive C such that for all positive h less than or equal to 1, we have the error between f'(x0) and its approximation by a central finite difference formula f(x0+h/2) - f(x0-h/2) over h, this error is now less than or equal to C*h². Same remark as before, C can depend on f and x0 because in our sentence C comes after "for all f", "for all x0" but C cannot depend on h because it appears before "for all positive h less than or equal to 1". So C can depend on f and x0 but not h. So the numerical interpretation is as follows : take f and x0 and take a look at the error when h varies the error, this quantity here, can be computed, so if we know f'(x0), we compute this quantity here and make the difference. This error is divided by 2²=4 every time h is divided by 2 or, if we divide h by 10 the error is divided by 10²=100. We need to prove this thoerem.

Notes

Summary

3m 05s



Proof: Be careful, as before we cannot choose C as follows 1/(8*6) times the absolute value of those derivatives here, that is C equal to 1/48 times the absolute value of f'''(ξ)+f'''(η) because this would depend on h, since ξ,η do depend on h, indeed ξ is between x0 and x0+h/2 and η between x0-h/2 and x0. But as before, we can take an upper bound with the maximum of the derivatives on the bigger interval. Here are x0, x0-h/2, x0+h/2. I use again the fact that h≤1, it is an arbitrary choice, we could have taken h ≤ 2, Thus x0-h/2 > x0-1/2 and x0+h/2 < x0+1/2. Now take the maximum of the third derivatives on this interval and we can finally show the theorem and choose C with an upper bound as the maximum of the third derivatives on this interval and so we have 1/24 because we have 1/48 but we're adding 2 terms so we have C=1/24 times the absolute value of the maximum of the 3rd derivative in the interval x0-1/2 and x0+1/2. Observe that C does depend on the 3rd derivatives of f and depends on x0 but not on h, so we proved the theorem.

Notes

Summary

5m 16s