Dérivées numériques d'ordre 1 – Formule de différences finies progressive (1)

Introduction à l'analyse numérique

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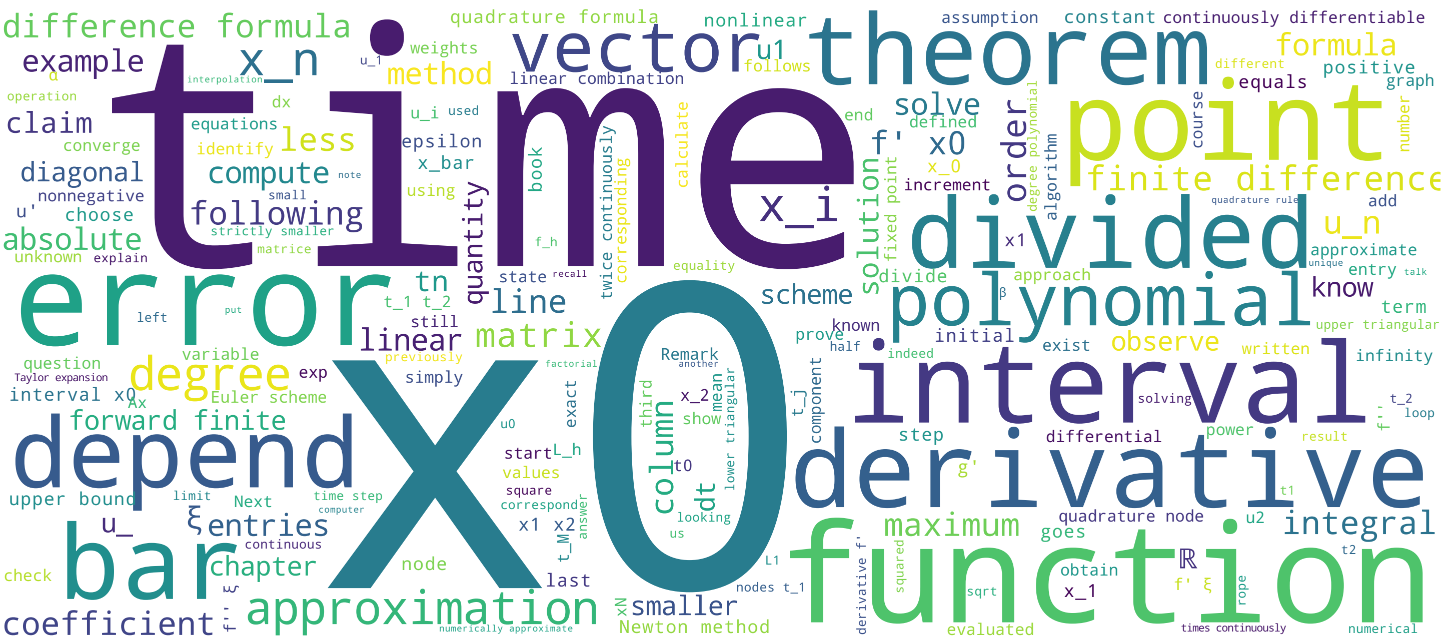


Video

Dérivée Num. d'ordre 1 : FDF progressive

$$\left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} \right| = O(h) ?$$

Soit $f : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^2$, soit $x_0 \in \mathbb{R}$, soit $h > 0$ Dev. Taylor : $f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(\xi)$ $\quad x_0 \leq \xi \leq x_0+h$

$$\left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} \right| = \frac{h}{2} \left| f''(\xi) \right|$$

Thm 2.2 : $\forall f \in \mathcal{C}^2$ $\forall x_0 \in \mathbb{R}$ $\exists C > 0$ $\forall 0 < h \leq 1$ on a $\left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h}\right.$

We'll now clarify the mathematical point of view. We want to show that the error between the derivative f'(x0) and its approximation by a forward finite difference formula is of order 1 in h. Let f be a function from ℝ to ℝ twice continuously differentiable. Take x0 in ℝ and take a fixed positive h. The Taylor expansion of f at x0 yields the following equality f(x0+h)=f(x0)+h*f'(x0)+h^2/2*f''(ξ) so we add h times the first derivative to f(x0) plus the square of the increment, h^2 over factorial of 2. so 2 times f'' at a certain point ξ. ξ is an indermediate point, between x0 and x0+h. From this equality we can easily deduce that f'(x0) minus f(x0+h)-f(x0) over h which is the approximation by the forward finite difference formula this is equal to h^2/2 times the absolute value of f'(ξ) Now let's state a mathematical theorem it is Theorem 2.2 of the book. I claim that for every function f, twice continuously diferentiable for all x0 in ℝ there is a positive C such that for all h positive less than or equal to 1 we get that the error, so f'(x0) minus its approximation by the forward finite difference formula f(x0+h)-f(x0) divided by h.

Notes

Summary

0m 03s

Dérivée Num. d'ordre 1 : FDF progressive

$$\left| f'(x_0) - \frac{f(x_0+h)-f(x_0)}{h} \right| = O(h) \ ?$$

Soit $f:\mathbb{R}\to\mathbb{R}$ $\mathcal{C}^2$, soit $x_0\in\mathbb{R}$, soit $h>0$ Dev. Taylor : $f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(\xi)$ $\quad x_0 \le \xi \le x_0+h$

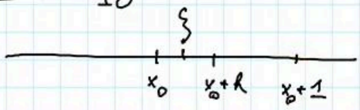
$$\left| f'(x_0) - \frac{f(x_0+h)-f(x_0)}{h} \right| = \frac{h}{2}\,|f''(\xi)|$$

Thm 2.2 : $\forall f\in\mathcal{C}^2\ \forall x_0\in\mathbb{R}\ \exists C>0\ \forall 0<h\le 1$ on a $\left| f'(x_0) - \frac{f(x_0+h)-f(x_0)}{h} \right| \le Ch$.

Rem : C dép de f, x_0 mais de h

Interprét : Choisit f, x_0 l'erreur est divisée par $\frac{2}{10}$ chaque fois que h est divisée par $\frac{2}{10}$

Dem : On ne peut pas choisir $C=\frac{1}{2}|f''(\xi)|$ mais car ξ dépend de h

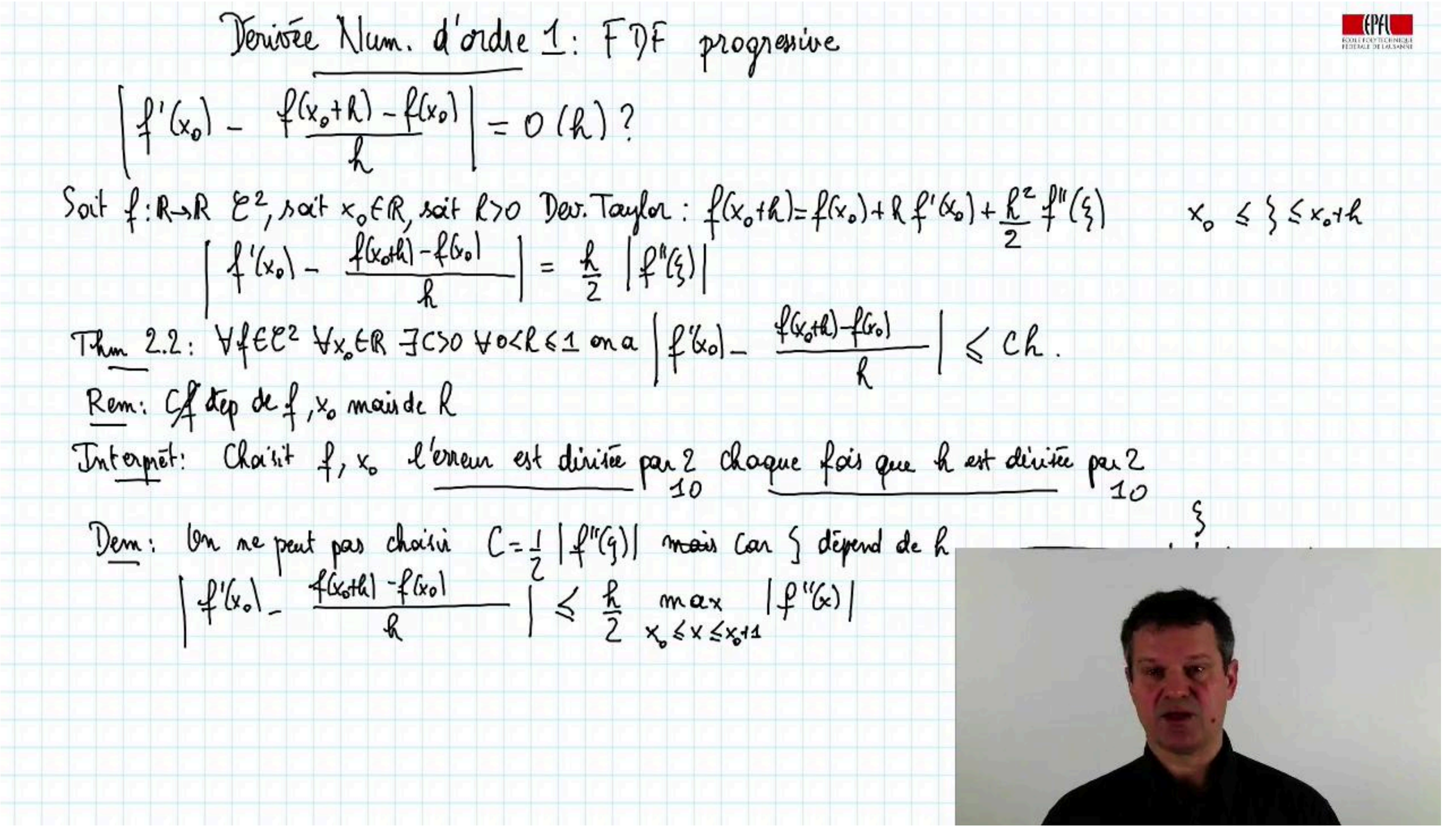


This quantity is less than C*h Remark that according to the theorem for all f and for all x0 there is a constant C, C depends only from what's before, that is f and x0 but C does not depend on what follows, so h hence C may depend on f and x0 but not on h. The numerical interpretation that we'll see is the following: Choose a function f and a point x0 and observe the error so that quantity. and let h vary. I claim that the error is divided by 2 every time h is divided by 2. Or the error is divided by 10 everytime we divide h by 10. We now have to prove the theorem. we may be tempted to choose C as 1/2 * f''(ξ), in absolute value, but we cannot, why? So be careful, we cannot take C = 1/2*f'(ξ), why ? Simply because ξ depends on h since it is between x0 and x0+h, so our C would depend on h. But what we can do is find an upper bound for the second derivative at ξ that will not depend on h. Let me explain more clearly. Here is the point x0, here you have x0+h ξ is somewhere between x0 and x0+h. Now I use the assumption that h is less than or equal to 1. so x0+h is less than or equal to x0+1 So now we take the upper bound for the second derivative at ξ as the maximum of the second derivative on the interval [x0,x0+1].

Notes

Summary

2m 21s



So I claim that f'(x0) minus its approximation by the forward finite difference formula (f(x0+h)-f(x0))/h but this time with an upper bound on the second derivative at ξ par le maximum des dérivées secondes on the interval [x0,x0+1], so it is less than or equal to h/2 times the maximum of the f''(x) in absolute value with all x between x0 and x0+1. So we get our constant C: C is half the maximum on the interval [x0,x0+1] of the second derivative. This C depends on f, more precisely on f'', and it depends on x0 because the interval for the maximum depends on x0, but C does not depend on h. So we proved the theorem.

Notes

Summary

5m 04s