Interpolation de Lagrange – cas n=2

Introduction à l'analyse numérique

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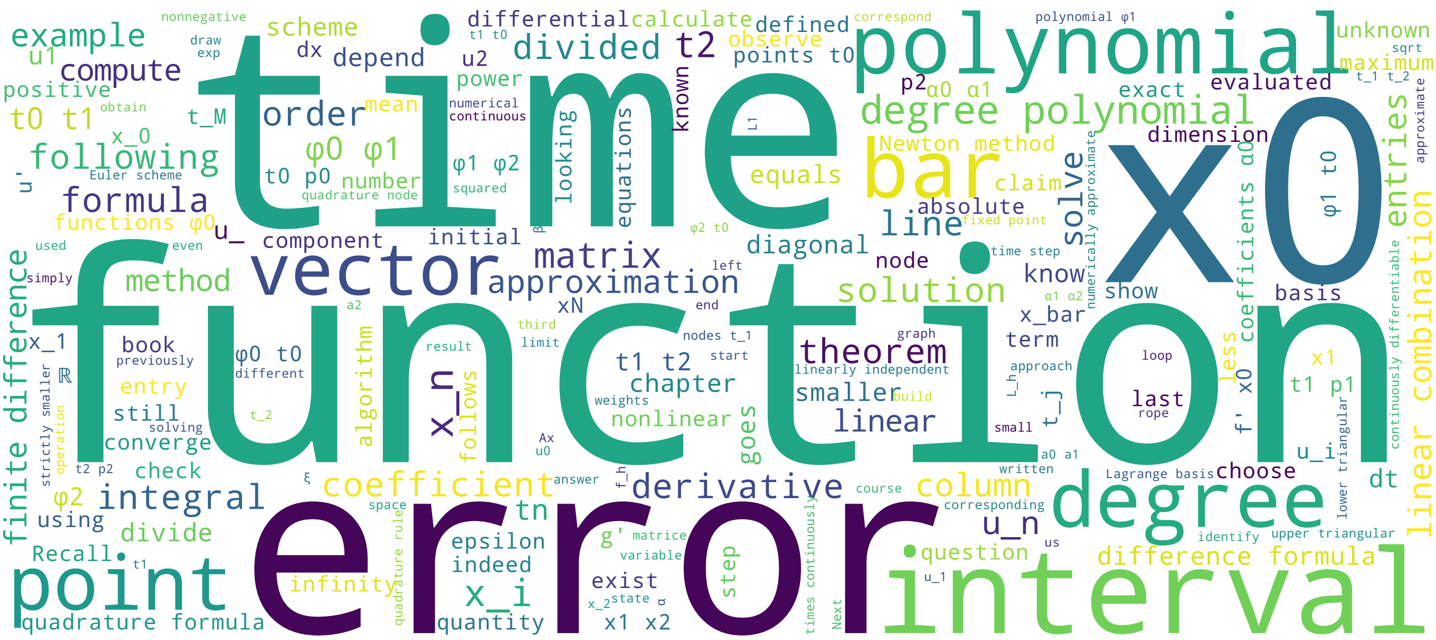
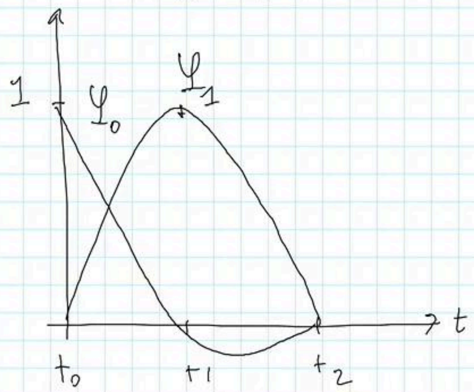


Video

EPFL

$n=2$: $t_0 \; t_1 \; t_2$ $\qquad \varphi_0 \varphi_1 \varphi_2$ base de Lagrange de $\mathbb{P}_2$ associée aux pts $t_0 \; t_1 \; t_2$

$\varphi_0 \varphi_1 \varphi_2 \in \mathbb{P}_2$



$\varphi_0 \varphi_1 \varphi_2$ base de $\mathbb{P}_2$:

$\varphi_0 \in \mathbb{P}_2 \quad \varphi_0(t_0)=1 \quad \varphi_0(t_1)=0 \quad \varphi_0(t_2)=0$

$$\varphi_0(t) = \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}$$

$\varphi_1 \in \mathbb{P}_2 \quad \varphi_1(t_0)=0 \quad \varphi_1(t_1)=1 \quad \varphi_1(t_2)=0$

$$\varphi_1(t) = \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)}$$

$\varphi_2 \in \mathbb{P}_2 \quad \varphi_2(t_0)=0 \quad \varphi_2(t_1)=0 \quad \varphi_2(t_2)=1$

Now we'll solve the problem with n=2. We have three pairwise distinct values t0, t1, t2 from which we'll construct φ0, φ1, φ2 the Lagrange basis for degree 2 polynomials, associated with t0, t1, t2. Those three functions φ0, φ1 and φ2 Are 3 polynomials of degree 2 built as follows : Here are our points t0, t1, t2 the first polynomial φ0 is a degree 2 polynomial such that φ0(t0) = 1, φ0(t1) = 0 and φ0(t2)=0 We can draw this φ0 and we can even write a formula for φ0 since it is zero at t1 and t2, so we can write it as (t-t1)*(t-t2) and I want it to be equal to 1 at t0 so I divide it by (t0-t1)*(t0-t2). So we get the formula for φ0 Likewise I can build the polynomial φ1, degree 2 polynomial such that φ1(t0)=0 φ1(t1)=1 φ1(t2)=0 So here is the polynomial φ1 and again we have a formula for φ1 which is (t-t0)*(t-t2) to get 0 at t0 and t2 and I want it to equal 1 at t1 so I divide it by (t1-t0)*(t1-t2). Finally we can build φ2 of degree 2 such that φ2(t0)=0 φ2(t1)=0 φ2(t2)=1 and we get an analogue formula. I now claim that those 3 functions φ0, φ1, φ2 are a basis for the set of polynomials of degree 2 or less. Indeed we have three functions in P2 the set of degree 2 polynomials.

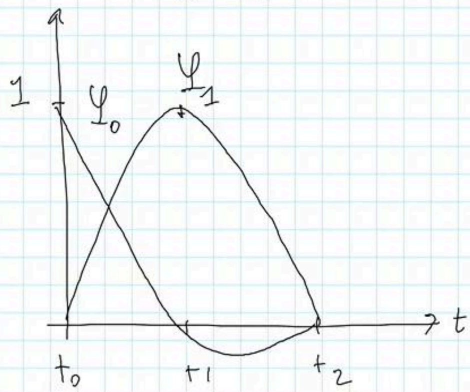
Notes

Summary

0m 00s

$n=2$: $t_0\ t_1\ t_2$ $\varphi_0\ \varphi_1\ \varphi_2$ base de Lagrange de $\mathbb{P}_2$ associée aux pts $t_0\ t_1\ t_2$

$\varphi_0\ \varphi_1\ \varphi_2 \in \mathbb{P}_2$



$\varphi_0 \in \mathbb{P}_2$ $\varphi_0(t_0)=1$ $\varphi_0(t_1)=0$ $\varphi_0(t_2)=0$

$$\varphi_0(t) = \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}$$

$\varphi_1 \in \mathbb{P}_2$ $\varphi_1(t_0)=0$ $\varphi_1(t_1)=1$ $\varphi_1(t_2)=0$

$$\varphi_1(t) = \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)}$$

$\varphi_2 \in \mathbb{P}_2$ $\varphi_2(t_0)=0$ $\varphi_2(t_1)=0$ $\varphi_2(t_2)=1$

$\varphi_0\ \varphi_1\ \varphi_2$ base de $\mathbb{P}_2$: dim $\mathbb{P}_2 = 3$ $p(t) \in \mathbb{P}_2$ $p(t) = a_0 + a_1 t + a_2 t^2$

linéairement indép : $\left(\alpha_0 \varphi_0(t) + \alpha_1 \varphi_1(t) + \alpha_2 \varphi_2(t) = 0 \quad \forall t \in \mathbb{R}\right) \Rightarrow \left(\alpha_0 = \alpha_1 = \alpha_2 = 0\right)$

$p \in \mathbb{P}_2$: $p(t) = \alpha_0 \varphi_0(t) + \alpha_1 \varphi_1(t) + \alpha_2 \varphi_2(t)$

Solution du pbm : $p(t) = p_0\, \varphi_0(t) + p_1\, \varphi_1(t) + p_2\, \varphi_2(t)$

The dimension of the space of degree 2 polynomials is 3. Why 3 ? Because if p is a degree 2 polynomial, write it a0+a1*t+a2*t^2. So {1, t, t^2} is the canonical basis for the degree 2 polynomials. There are 3 elements in this basis so the dimension of P2 is 3 Hence the only thing to check to get that φ0, φ1, φ2 are a basis is that those three functions are linearly independent, so I take 3 coefficients α0, α1, α2 and I write the linear combination α0*φ0+α1*φ1+α2*φ2 and suppose this equals 0. I have to show that this implies that all coefficients α0, α1, α2 are 0. To show this, it suffices to take t=t0 to get α0=0 t=t1 to get α1=0 and t=t2 to get α2=0. so those 3 functions are linearly independent which means that if p is a degree 2 polynomial, we can write p(t) as a linear combination if those 3 functions α0φ0(t) +α1φ1(t) +α2φ2(t) Now I will give the solution of the problem. Recall I am looking for a degree 2 polynomial that goes through the points (t0,p0), (t1,p1), (t2,p2). The solution is just a linear combination of φ0, φ1 and φ2, and the coefficients of this linear combination are p0, p1 and p2. So p(t)=p0*φ(t)+p1*φ(t)+p2*φ(t). We can easily check that this is the solution to our problem.

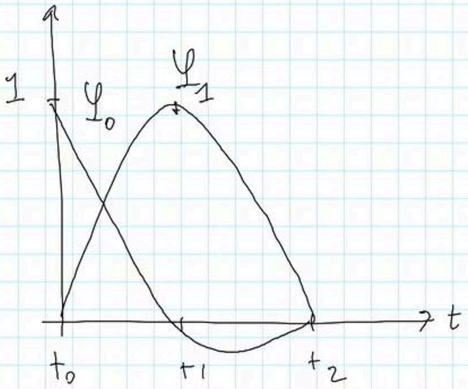
Notes

Summary

3m 12s

$n=2$: $t_0\ t_1\ t_2$ $\quad$ $\varphi_0\varphi_1\varphi_2$ base de Lagrange de $\mathbb{P}_2$ associée aux pts $t_0\ t_1\ t_2$

$\varphi_0\ \varphi_1\ \varphi_2 \in \mathbb{P}_2$



$\varphi_0 \in \mathbb{P}_2$ $\quad \varphi_0(t_0)=1 \quad \varphi_0(t_1)=0 \quad \varphi_0(t_2)=0$

$$\varphi_0(t)=\frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}$$

$\varphi_1 \in \mathbb{P}_2$ $\quad \varphi_1(t_0)=0 \quad \varphi_1(t_1)=1 \quad \varphi_1(t_2)=0$

$$\varphi_1(t)=\frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)}$$

$\varphi_2 \in \mathbb{P}_2$ $\quad \varphi_2(t_0)=0 \quad \varphi_2(t_1)=0 \quad \varphi_2(t_2)=1$

$\varphi_0\ \varphi_1\ \varphi_2$ base de $\mathbb{P}_2$: dim $\mathbb{P}_2=3$ $\quad p(t)\in\mathbb{P}_2 \quad p(t)=a_0+a_1t+a_2t^2$

linéairement indép : $\left(\alpha_0\varphi_0(t)+\alpha_1\varphi_1(t)+\alpha_2\varphi_2(t)=0 \quad \forall t\in\mathbb{R}\right) \Rightarrow \left(\alpha_0=\alpha_1=\alpha_2=0\right.$

$p\in\mathbb{P}_2$: $p(t)=\alpha_0\varphi_0(t)+\alpha_1\varphi_1(t)+\alpha_2\varphi_2(t)$

Solution du pbm : $p(t)=p_0\varphi_0(t)+p_1\varphi_1(t)+p_2\varphi_2(t) \quad \in\mathbb{P}_2$

$p(t_0)=p_0\cdot 1 \quad + p_1\cdot 0 \quad +$

p is a degree 2 polynomial, because it is a linear combination of those 3 functions, φ0, φ1 and φ2. And one can check that for example p(t0)=p0. Why ? Because φ0(t0)=1 and then p1*φ1(t0)=0. and p2*φ2(t0)=0. so we indeed have p(t0)=p0. Likewise p(t1)=p1 and p(t2)=p2.

Notes

Summary

5m 51s