





- Lois de Kepler
- Equations du mouvement
- Energie
- Trajectoire (ellipse)
- Loi de la gravitation

Mécanique | 2013 7

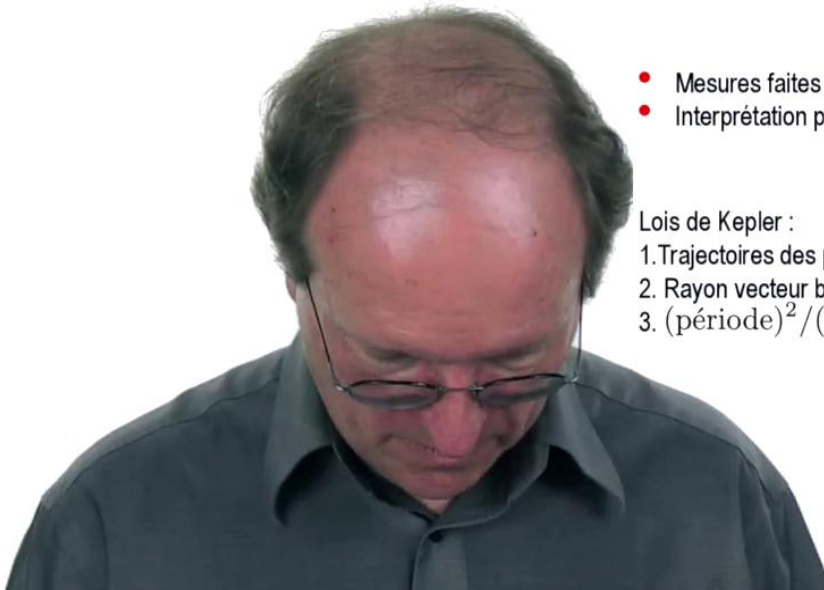
Hello, welcome to the ÉPFL General Physics course. In this lesson, I will introduce a great classic of mechanics: the problem of planets around the Sun. We will see, here, that we can conduct an analysis, in which we know the motion, we know the characteristics of the motion, and we can deduce the force. Usually, in what we have done so far, in particular, we have given ourselves a force model, we have written the equations of motion, we have integrated them to obtain the motion. Here, we can start from the motion to find the force. I'll start by stating Kepler's laws, which characterize the motion of the planets around the Sun. We will write the equations of motion, by giving ourselves a force model with a dependence in  $1$  on the distance squared. We will see that the energy is conserved. This will be a particular case that we have already seen; We have seen in a general way, that when we have forces that are conservative, we have the conservation of energy. We will see it here, by deducing from the equations of motion. We will discuss the trajectory of the planets around the Sun, obtain the result that the orbits are ellipses. And finally, we will, with Kepler's laws, obtain Newton's law of gravitation.

Notes

Summary



0m 04s



- Mesures faites par Tycho Brahé (1546 -1601)
- Interprétation par Kepler (1571 - 1630)

Lois de Kepler :

1. Trajectoires des planètes : ellipses, Soleil au foyer
2. Rayon vecteur balaie des aires égales en un temps fixé
3.  $(\text{période})^2 / (\text{grand axe})^3$  pour toutes les planètes

Mécanique | 2013 10

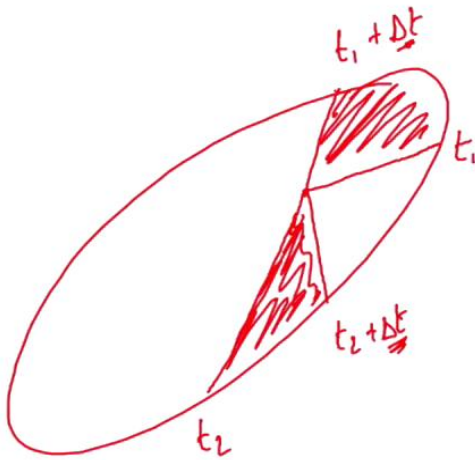
I start with Kepler's laws. The story goes back to the sixteenth century, when Tycho Brahe made extremely precise measurements, and quite remarkable, with the instrumentation he had developed at the time, extremely precise measurements, of the orbits of the planets, of a few planets orbiting the Sun. Kepler, a student of Tycho Brahe, examined these data very carefully, and deduced the following laws: He observed, on the one hand, that the planets describe elliptical orbits, and that the Sun is at the focus of the ellipse. On the other hand, he states what is called the law of areas, which says that the vector ray sweeps equal areas, in a fixed time. I have to draw a picture for this.

Notes

Summary



1m 47s



- Mesures faites par Tycho Brahé (1546 -1601)
- Interprétation par Kepler (1571 - 1630)

Lois de Kepler :

1. Trajectoires des planètes : ellipses, Soleil au foyer
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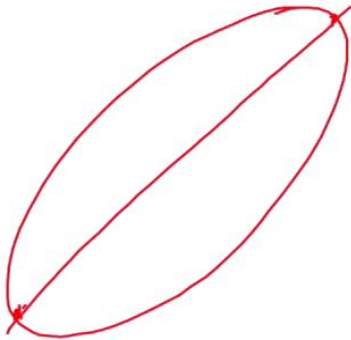
Mécanique | 2013 10

So, I have, you imagine, a very eccentric orbit. This is supposed to represent an ellipse. This is the focus. So, I consider the vector radius, say, between time  $t_1$ , and time  $t_1$  plus some  $\Delta t$ . The OP vector ray sweeps through that area. Then I take a time later, say, like this. I take a time  $t_2$ , and a  $t_2$  plus the same  $\Delta t$ . The same  $\Delta t$ . And what the law of areas tells us is that the vector ray swept the same areas. This is that law.

Notes

Summary





- Mesures faites par Tycho Brahé (1546 -1601)
- Interprétation par Kepler (1571 - 1630)

Lois de Kepler :

1. Trajectoires des planètes : ellipses, Soleil au foyer
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Kepler's third law tells us, then, that the period is the time it takes to make one orbit, divided by the major axis, that, that length, is what we call the major axis. So, the period squared divided by the major axis cube, has the same value for all the planets.

Notes

Summary

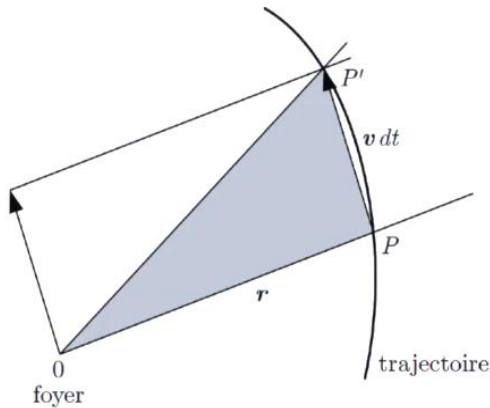


3m 44s

$$dA = \text{aire}(OPP') = \frac{1}{2} r v dt \sin(\mathbf{r}, \mathbf{v})$$

Mouvement dans un plan :

$\mathbf{r} \wedge \mathbf{v}$  toujours dans la même direction.



Mécanique | 2013 15

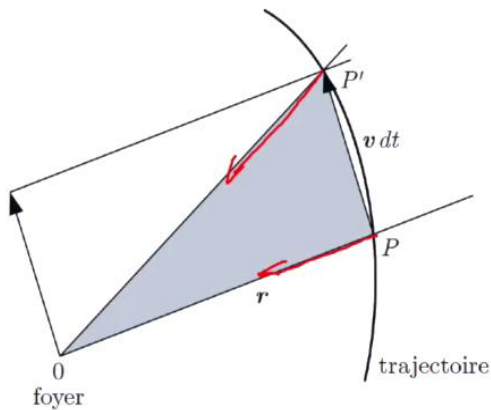
So let's start by analyzing what the law of areas means. I imagine, here, an element of the trajectory, the focus, the OP vector, the ray vector that sweeps an area. So, I will take the displacement between a time  $t$  and  $t$  plus  $dt$ . So, I have an infinitesimal displacement. I have an infinitesimal area, and to calculate that area, which I'll call  $da$ , it must be, (it's equal to) the area of the triangle O P P prime. Which I've written down, here. So, that's half the base times the height, formula for the area of a triangle. So, I'm doing half of  $r$  times the height of triangle. Then, I would have to calculate the *sinus* of the angles, here. Here, I have practically drawn a 90 degree angle. But this angle here, this is the one that comes into play. And so, I have  $vdt$  times this *sinus*, which gives me the height of the triangle. And there you have it, the formula for the area. Now, since we have planar motion,  $r$  and  $v$  are still in the same plane. So,  $r$  cross  $v$  is always along the normal to the plane. And note that this formula there, you have  $r$ ,  $v$ , times the *sinus* of the angle between  $r$  and  $v$ .

Notes

Summary



4m 15s



$$dA = \text{aire}(OPP') = \frac{1}{2} r v dt \sin(\mathbf{r}, \mathbf{v})$$

Mouvement dans un plan :

$\mathbf{r} \wedge \mathbf{v}$  toujours dans la même direction.

1<sup>ère</sup> et 2<sup>ème</sup> loi de Kepler :

$$\mathbf{A} = \frac{1}{2} \mathbf{r} \wedge \mathbf{v} = \frac{1}{2m} \mathbf{L}_O \quad \text{« Vitesse aréolaire »}$$

Moment cinétique constant :

$$\frac{d}{dt} (\mathbf{r} \wedge \mathbf{v}) = 0 = \mathbf{v} \wedge \mathbf{v} + \mathbf{r} \wedge \frac{d\mathbf{v}}{dt} = \mathbf{r} \wedge \mathbf{a} \implies \text{Force « centrale »}$$

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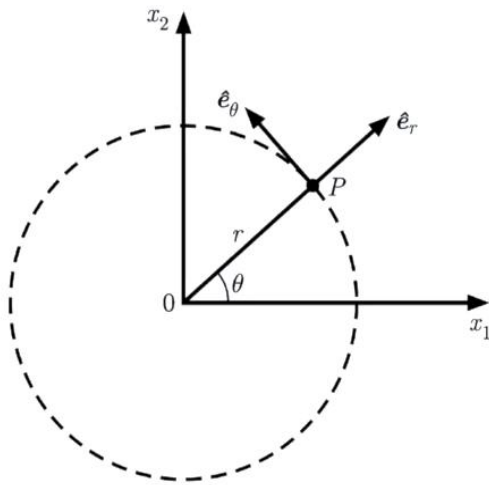
This is indeed the modulus of that vector. Therefore, Kepler's first law and second law together, tell us that the vector  $\mathbf{I}$  can construct as follows. I write  $\mathbf{r} \wedge \mathbf{v}$ , with the one-half, which has a fixed direction. And the law of areas tells us that the modulus is constant too. That vector there is a constant. Now, we recognize here, a factor of  $2m$ , the angular momentum. We would have  $\mathbf{r} \wedge m\mathbf{v}$ , for angular momentum. This, we call the areal velocity. Now, if angular momentum is conserved,  $\mathbf{r} \wedge m\mathbf{v}$  is conserved, That means its derivative with respect to time is zero. There are two terms; The first term, obviously, gives 0. The second term,  $\frac{1}{2} \frac{d\mathbf{v}}{dt}$  That means  $\mathbf{A}$ . And  $\mathbf{A}$  is  $\frac{1}{2m} \mathbf{L}$ . So, we are seeing that, here, this law of areas implies that  $\mathbf{r}$  and  $\mathbf{F}$  are collinear :  $\mathbf{r}$  is parallel to  $\mathbf{F}$ . We say that the force is central. This means that, at all times, the force is directed toward the focus, hence, toward the Sun.

Notes

Summary



5m 54s



Référentiel : un groupe d'étoiles

Coordonnées 'polaires' (cylindrique dans le plan)

$$\mathbf{r} = r \mathbf{e}_r$$

$$\mathbf{v} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta$$

$$\mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \mathbf{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \mathbf{e}_\theta$$

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I will now establish the kinematics of this motion. I give myself, as reference, a group of stars, distant stars. I will use coordinates that I call polar, it's simply cylindrical coordinates restricted to the plane. I'll use the following notation: I take axes related to the referential, I define an angle  $\theta$  with respect to the  $x_1$  axis. Here, this circle is the coordinate line where  $\theta$  varies.  $r$ , this is going to be the distance, this is the equivalent of  $\rho$  for cylindrical coordinates, I have a vector  $\mathbf{e}_r$ , unit vector  $\mathbf{e}_\theta$ , unit vector that forms itself the reference frame associated with its coordinates. I have well known results. The vector, the radius vector  $\mathbf{OP}$ , I note  $\mathbf{r}$ , it is simply the coordinate  $r$  times  $\mathbf{e}_r$ . The velocity has a  $\dot{r}$  point term, and a  $r \dot{\theta}$  point term, and the acceleration, I take from the form, adjusting the notation, so, the form for the cylindrical coordinates.

Notes

Summary



7m 37s





$$L_O = m r \underbrace{e_r}_{\vec{r}} \wedge \underbrace{(\dot{r}e_r + r\dot{\theta}e_\theta)}_{\vec{v}} = m r^2 \dot{\theta} e_z = L e_z$$

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Now, let's express the angular momentum in polar coordinates. So, I need to write the vector  $r$  cross  $mv$ . The  $m$  is there. And there we have  $v$  in polar coordinates. The first term gives 0. The second term gives  $m, r$  square  $\dot{\theta}$ , in the  $e_r$  cross  $e_\theta$  direction, what I'll call  $e_z$ . This is the unit vector, normal to the plane.

Notes

Summary



8m 53s



$$\mathbf{L}_O = m r \mathbf{e}_r \wedge (\dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta) = m r^2 \dot{\theta} \mathbf{e}_z = L \mathbf{e}_z$$

$$\boxed{L = m r^2 \dot{\theta}} = \text{constante}$$

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And I'm going to denote  $L$  this magnitude, magnitude  $m r^2 \dot{\theta}$  point,  $L$  is a constant, let's not forget that. And we'll often use, this constant of motion that gives a link between  $r$  and  $\theta$  point.

Notes

Summary



9m 27s

# Equation du mouvement

Modèle de force

On procède en posant déjà :

$$\mathbf{F} = -\frac{K}{r^2} \mathbf{e}_r \quad (K > 0)$$

$$m(\ddot{r} - r\dot{\theta}^2) = \frac{-K}{r^2}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \longrightarrow mr^2\ddot{\theta} + 2mrr\dot{\theta} = \frac{d}{dt} \left( \underbrace{mr^2\dot{\theta}}_L \right) = 0$$

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Now let's write the equations of motion. Then, strictly speaking, we could start from Kepler's laws, and show that the force being 1 on  $r$  square. To keep things more explicit, we'll give ourselves a model of force as 1 over  $r$  square. I'll take  $K$  positive. The force is opposite to  $\mathbf{e}_r$ . So it is directed towards the center. So it is an attractive force, which I assume, and with a dependence in 1 on  $r$  square. I put this down to develop my calculations. The equation of motion, here, we have the two components of acceleration, according to  $\mathbf{e}_r$  and according to  $\mathbf{e}_\theta$ . There is a force, only according to  $\mathbf{e}_r$ , because we have shown that we have a central force. And now I begin my analysis. The first thing I'd like to point out to you, is that the second equation, I can multiply it by  $r$ . I have those two terms. And, if you look at it a little closer, you see that here, you have the derivative with respect to time of  $mr^2\dot{\theta}$  point. And this is what we had called  $L$ , the angular momentum, or the modulus of angular momentum. So, this equation of motion implies the conservation of angular momentum.

Notes

Summary



9m 52s

# Equation du mouvement

Modèle de force

On procède en posant déjà :

$$\mathbf{F} = -\frac{K}{r^2} \mathbf{e}_r \quad (K > 0)$$

$$L = mr^2\dot{\theta}$$

$$m(\ddot{r} - r\dot{\theta}^2) = \frac{-K}{r^2}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \longrightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$m(\dot{r}\ddot{r} - r\dot{\theta}^2) = m(\dot{r}\ddot{r} - \frac{L^2\dot{r}}{m^2r^3}) = \frac{-K\dot{r}}{r^2}$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{K}{r}\right) = 0$$

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I put the value of L here, because now, what I propose to do is to take that equation of motion, multiply it by  $r$  point. You see that you have  $r$  point appearing, here, here, here, and then here. And here, what I've done is I've used this L to replace theta point squared, with a function of  $r$ . Theta point is L over  $mr$  square. So, theta point squared is L squared over  $m$  squared  $r^4$ . There is an  $r$ , here, so there is only an  $r$  cube left. And now I'm contemplating this equation here, and I'm thinking "Aha!" I recognize derivatives with respect to time here, for example,  $r$  point,  $r$  point point, that's the derivative with respect to time of  $r$  point squared. That's what I noted here. Here, we have minus 1 on  $r$  cube, that's the derivative of 1 on  $r$  square; indeed, if we take that term there and derive it with respect to time, we'll have a minus 2, which simplifies with that 2,  $r$  to the power minus 3, times  $r$  dot; the  $r$  dot it is there. And then that term there, if I drift it with respect to time, I'm going to have plus 1 on  $r$  square, times  $r$  dot, I pass it to the other side of the sign, the equal sign, so I have a minus  $r$  dot on  $r$  square. So, I found that this term here is a constant of motion.

Notes

Summary



11m 20s

Constante du mouvement :

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{m r^2} - \frac{K}{r} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{K}{r}$$

$$\mathbf{F} = -\frac{K}{r^2} \mathbf{e}_r \quad V(r) = \int_r^\infty -\frac{K}{r'^2} dr' = K \left[ \frac{1}{r'} \right]_r^\infty = \frac{-K}{r}$$

$\frac{-K}{r}$  est l'énergie potentielle.

$E$  est l'énergie mécanique.

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I summarize: I'm going to call this term E, it's a constant of motion, I call it energy because I recognize here a kinetic energy, so, by consistency of units, all these terms here are energy terms; if now, instead of writing L square, I write r square theta dot square, I recognize right away here the r component of v squared, and here the theta component of v squared. So I do have the kinetic energy here, and this must be the potential energy. We can check it: if this is the force, the potential energy is the work to get from this r position, to a reference position and here I'm going to take the reference to infinity; so I mean the zero of the potential will be at infinity. I calculate the work of the force, and that integral there gives me 1 over r, so r prime, the variable of integration, I call it r prime, which I want to calculate at the r position and at infinity, it gives me that term there. So the minus K on r, is indeed the potential energy, and the E, is indeed the mechanical energy.

Notes

Summary

13m 05s



$$m(\ddot{r} - r\dot{\theta}^2) = \frac{-K}{r^2} \quad L = mr^2\dot{\theta}$$

Changement de variable :

$$q = \frac{1}{r} \quad \dot{r} = \frac{-1}{q^2} \frac{dq}{d\theta} \dot{\theta} = -r^2 \dot{\theta} \frac{dq}{d\theta} = -\frac{L}{m} \frac{dq}{d\theta}$$

$$\ddot{r} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \dot{\theta} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \frac{L}{mr^2} = -\frac{L^2}{m^2} q^2 \frac{d^2q}{d\theta^2}$$

Now we move on to the question of trajectories. I start from this equation of motion, without forgetting this constant of motion, the angular momentum. I make a change of variable, so there, I have to point out to you that the passage I'm going to show you now is a passage which is ad hoc, which concerns this question of Kepler's problem, it should not be seen as a skill to be developed; but let's look at what happens if we make this change of variable. I need to calculate  $r$  point point, so we'll go one step at a time, we'll calculate  $r$  point first. So here, what I'm doing is I'm considering  $r$  to be 1 on  $q$ ,  $q$  is a function of theta, and theta is a function of  $t$ . So, to calculate the derivative, I have the derivative of 1 on  $q$ , that's minus 1 on  $q$  squared, you have to derive  $q$  with respect to theta, and you have to derive theta with respect to time, that's theta point. Now, 1 over  $q$  square, that's  $r$  square,  $r$  square theta point, that's  $L$  over  $m$ , that's what I wrote here. Now, we can calculate  $r$  dot dot. So, starting from there, we have  $d$  square  $q$  over  $d$  theta square times theta point.

Notes

Summary



$$m(\ddot{r} - r\dot{\theta}^2) = \frac{-K}{r^2} \quad L = mr^2\dot{\theta}$$

Changement de variable :

$$q = \frac{1}{r} \quad \dot{r} = \frac{-1}{q^2} \frac{dq}{d\theta} \dot{\theta} = -r^2 \dot{\theta} \frac{dq}{d\theta} = -\frac{L}{m} \frac{dq}{d\theta}$$

$$\ddot{r} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \dot{\theta} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \frac{L}{mr^2} = -\frac{L^2}{m^2} q^2 \frac{d^2q}{d\theta^2}$$

$$\frac{d^2q}{d\theta^2} + q = \frac{Km}{L^2}$$

Theta point, we can write it as  $L$  over  $mr$  square, or  $L$  over  $m$ , times  $q$  square. Hence the  $q$  square that is here. And now we need to put this  $r$  dot dot, in there. So here we go:  $m r$  dot dot, that'll be less  $L$  square over  $m q$  square, let me call these terms there,  $q$  second. Here I have minus  $m r$ ,  $r$  is  $1$  over  $q$ , theta point square, it's  $L$  square over  $m$  square,  $r^4$ , that's  $q^4$ . And all of this is equal to minus  $K$  times  $q$  square. Now we can change the color, we'll divide by  $q$  square. There I have a  $q$  cube, there will be only  $q$  left, and now I'm dividing by  $q$  square. I'm going to put the  $L$  square here, so I take it out of there, and I take it out of there. I've got a  $m$  that I'm going to bring back here, so I'm removing these two and this one, and there's minus signs all over the place, which I'm removing. And I have the equation of motion in  $q$ , which looks like this:  $d$  square  $q$  over  $d$  theta square, plus  $q$ , equals  $Km$  over  $L$  square. Now, I claim that this equation there, this equation of motion, this differential equation, you already know it.

Notes

Summary



$$m(\ddot{r} - r\dot{\theta}^2) = \frac{-K}{r^2} \quad L = mr^2\dot{\theta}$$

Changement de variable :

$$q = \frac{1}{r} \quad \dot{r} = \frac{-1}{q^2} \frac{dq}{d\theta} \dot{\theta} = -r^2 \dot{\theta} \frac{dq}{d\theta} = -\frac{L}{m} \frac{dq}{d\theta}$$

$$\ddot{r} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \dot{\theta} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \frac{L}{mr^2} = -\frac{L^2}{m^2} q^2 \frac{d^2q}{d\theta^2}$$

$$\frac{d^2q}{d\theta^2} + q = \frac{Km}{L^2} \quad q = \frac{1}{r} = \frac{Km}{L^2} + C \cos(\theta + \theta_0) \quad \text{Equation polaire d'une ellipse}$$

Valeurs extrémales :

$$\frac{1}{r_1} = \frac{Km}{L^2} + C \quad \frac{1}{r_2} = \frac{Km}{L^2} - C$$

I invite you to pause, to think and remember where you've encountered it before. So, this equation here is the equation of a harmonic oscillator, where  $q$  would represent the position of the mass,  $\theta$  plays the role of time, and instead of having a  $K$  on  $m$  here, we have 1. And here, we have a constant term, just like we had a constant term when we had a spring, a mass attached to a spring, in the gravity field. So, we know the solution of this equation. The solution for  $q$  is the constant term, plus a harmonic term; here the pulsation is 1, because here I had 1. I have two constants of motion,  $\theta_0$ , I mean, I have two constants of integration,  $\theta_0$  and  $C$ , which are to be determined by the initial conditions. Now, I invite you to consult a math book, a mathematician buddy, or do some calculations, you can convince yourself that this equation here, 1 on  $r$  function of  $\theta$  like this, is the equation in polar coordinates of an ellipse. There is no point in the mechanics course in looking at this further, there is one small result I need, which is that if I want to find the extremal values of  $r$ , the largest value of  $r$  and the smallest value of  $r$ , I have to take the largest value of cosine  $\theta$  and the smallest value of cosine  $\theta$ , of the other term.

Notes

Summary





$$m(\ddot{r} - r\dot{\theta}^2) = \frac{-K}{r^2} \quad L = mr^2\dot{\theta}$$

Changement de variable :

$$q = \frac{1}{r} \quad \dot{r} = \frac{-1}{q^2} \frac{dq}{d\theta} \dot{\theta} = -r^2 \dot{\theta} \frac{dq}{d\theta} = -\frac{L}{m} \frac{dq}{d\theta}$$

$$\ddot{r} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \dot{\theta} = -\frac{L}{m} \frac{d^2q}{d\theta^2} \frac{L}{mr^2} = -\frac{L^2}{m^2} q^2 \frac{d^2q}{d\theta^2}$$

$$\frac{d^2q}{d\theta^2} + q = \frac{Km}{L^2}$$

$$q = \frac{1}{r} = \frac{Km}{L^2} + C \cos(\theta + \theta_0)$$

Equation polaire d'une ellipse

Valeurs extrémales :

$$\frac{1}{r_1} = \frac{Km}{L^2} + C \quad \frac{1}{r_2} = \frac{Km}{L^2} - C \quad (C < Km/L^2)$$

If I draw a picture, you have, here is the focus, here you have the smallest value of  $r$ , this is  $r_1$  and here you have the largest, this is  $r_2$ . So what have I done here? It's that I've taken, for  $r_1$ , I've taken cosine is 1, and then for  $r_2$ , the largest value, I have to take the smallest possible value here is obviously when the cosine is minus 1. As  $r_2$  is a distance to the focus,  $r_2$  is a positive quantity, so there is a condition which is imposed on  $C$ , there is no need to look at that more in detail here, but indeed, we have an ellipse, only if we launch our object around the Sun, with a speed not too great. This condition depends on the attraction force, the mass of the object which we launch into the orbit and the angular momentum.

Notes

Summary



# 3ème loi de Kepler

(vitesse aréolaire)  $\times$  (période) = aire de l'ellipse

$$\frac{L}{2m}T = \text{aire}$$

$$\text{aire} = \int_0^T \frac{L}{2m} dt = \int_0^T \frac{1}{2} r^2 \dot{\theta} dt = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{(KM/L^2 + C \cos \theta)^2} = \frac{1}{C^2} \frac{\pi (Km/CL^2)}{((KM/CL^2)^2 - 1)^{3/2}}$$

Grand axe :  $2a = \frac{1}{\frac{Km}{L^2} - C} + \frac{1}{\frac{Km}{L^2} + C}$

Now we exploit Kepler's third law. Then we have the areal velocity, which is constant. If we multiply it by the period, we should get the area of the ellipse. The areal velocity is  $L$  over  $2m$ , the period, I'll call it  $T$ , and so this product there must give the area of the ellipse. Now, we can reason in another way to get the area of the ellipse, we can do the integral from 0 to  $T$  of the areal velocity, times  $dt$ ; that term I write as  $r$  square, theta point. Now I do a change of variable, I use theta as a variable, theta varying from 0 to  $2\pi$ , and I have half of  $r$  square  $d$  theta. With the  $r$  that is given by this  $r$  of theta, which is therefore the result that we got for the ellipse; shown here. This integral, a computer, an integration program knows how to do it algebraically, that's how I found the solution myself, I put this formula there in an integrator, and here is the result I got. Well, now, you remember that Kepler's law says, invokes, the period squared is the major axis cube. So the major axis, we'll calculate it, we'll call it  $2a$ , we'll calculate it as,  $r_1$  plus  $r_2$ .

Notes

Summary



# 3<sup>ème</sup> loi de Kepler

(vitesse aréolaire)  $\times$  (période) = aire de l'ellipse

$$\frac{L}{2m}T = \text{aire}$$

$$\text{aire} = \int_0^T \frac{L}{2m} dt = \int_0^T \frac{1}{2} r^2 \dot{\theta} dt = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{(KM/L^2 + C \cos \theta)^2} = \frac{1}{C^2} \frac{\pi (Km/CL^2)}{((KM/CL^2)^2 - 1)^{3/2}}$$

Grand axe :  $2a = \frac{1}{\frac{Km}{L^2} - C} + \frac{1}{\frac{Km}{L^2} + C}$

$$\frac{T^2}{a^3} = \frac{4\pi^2 m}{K}$$

So  $r_1$  and  $r_2$ , I gave them earlier, here they are, expressed here, and now we have to do some algebra, it's not very interesting, to simplify that term, use that term there and that result there for the area, to get the  $T$  square, on  $a$  cube, which appears in Kepler's law, Kepler's third law. And everything simplifies magnificently, only these terms remain, there remains  $4\pi^2 m$  over  $K$ . There's no reason to spend time looking at these algebraic details, we're not going to learn mechanics by doing this.

Notes

Summary





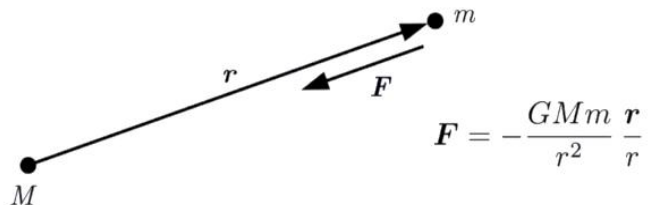
$$\frac{T^2}{a^3} = \frac{4\pi^2 m}{K}$$

Newton :  
si c'est vrai pour toute planète :

$$K \propto m$$

Newton :  
le soleil joue un rôle semblable à celui de la planète

$$K \propto M m$$



$$G = 6.7 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

Constante gravitationnelle

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So, now, it becomes much more interesting, because we just found this  $T^2$  square on  $a^3$  cube, and Kepler, Kepler's third law tells us that this  $T^2$  square term on  $a^3$  cube, is the same for all planets; now, planets have different masses, so, we absolutely have to take  $K$  proportional to the mass  $m$  of the planet. And there Newton makes a rather beautiful reasoning, he says: If this force that I am trying to characterize, depends on the mass of the planet, by symmetry, this force must also depend on the mass of the Sun. So, he says  $K$  proportional to the two masses, the mass of the Sun and the mass of the planet. And he writes, well, and so I put it here geometrically, you have the Sun here, the planet, the radius vector, I define a unit vector in this way. And I have force in  $K$  over  $r^2$ , where  $K$  is a constant, times the two masses. This constant is called the gravitational constant, I give you its value here, and so, we have concluded the analysis of this problem.

Notes

Summary



23m 30s