

Energie de l'oscillateur harmonique



- Energie
- Dissipation
- Facteur de qualité

Mécanique | 2013 5

Hello, welcome to the EPFL general physics course. In this lesson, I introduce the notion of potential energy and conservation of energy. We will now see what it looks like when we look at a harmonic oscillator. So, I will first calculate the energy of the free harmonic oscillator and then a harmonic oscillator with friction. Then we'll look at what in the technique we call the *quality coefficient*.

Notes

Summary



0m 04s

Energie, sans amortissement



Oscillateur harmonique sans amortissement :

$$x = C \cos(\omega_0 t + \Phi)$$

Mécanique | 2013 7

I start with the energy of a harmonic oscillator when there is no friction.
So I guess I have something like this.

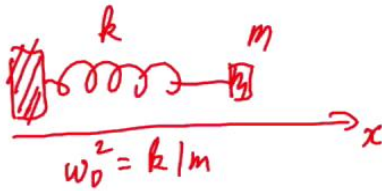
Notes

Summary



0m 40s

Energie, sans amortissement



Oscillateur harmonique sans amortissement :

$$x = C \cos(\omega_0 t + \Phi)$$

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You have a support that belongs to the frame of reference, a spring, a mass and I have a coordinate x that describes the motion of this object. So I found that in the absence of damping, x has the form of a cosine with the ω_0 that was worth root of k over m when here I have k and ω_0 squared that is worth k over m . A little reminder. So here's the x , and now I'll calculate the energy.

Notes

Summary



0m 53s



Oscillateur harmonique sans amortissement :

$$x = C \cos(\omega_0 t + \Phi)$$

Energie cinétique :

$$\dot{x} = -C\omega_0 \sin(\omega_0 t + \Phi)$$

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m C^2 \omega_0^2 \sin^2(\omega_0 t + \Phi)$$

Energie potentielle :

$$V(x) = \frac{1}{2} k x^2 = \frac{1}{2} m \omega_0^2 C^2 \cos^2(\omega_0 t + \Phi)$$

Energie mécanique :

$$E = \frac{1}{2} m C^2 \omega_0^2 = \text{constante}$$

Mécanique | 2013 13

First I'm going to calculate the kinetic energy, I need the velocity, the derivative of the cosine is going to give me *minus* [-] the sine with an omega 0 I'm calculating the kinetic energy, half of mx point *squared* [²]. Then I have the sine square and a C square omega 0 square. Now for the potential energy we saw that it was worth half of k x square. Now for k, I'll write "m omega 0 square". Again I can write "k = m omega 0 square". That's what I did here. And now I calculate the total energy which is the kinetic energy plus the potential energy. I have the same coefficients in front of the sine squares, cos squares, sum of the sin squares cos squares gives 1. I still have the mechanical energy which is half of m C square omega zero square. I calculate T+V. I have the same coefficients here and sin square cos square which gives one. So here is my energy and I found that the energy was indeed a constant, as advertised.

Notes

Summary



1m 36s

Energie, avec amortissement faible

Oscillateur harmonique faiblement amorti :

$$\begin{aligned}x &= e^{-\gamma t} C \cos(\omega_1 t + \Phi) & \gamma &\ll \omega_0 \\ \dot{x} &= C e^{-\gamma t} \{-\gamma \cos(\omega_1 t + \Phi) - \omega_1 \sin(\omega_1 t + \Phi)\} \\ &\cong C e^{-\gamma t} \{-\omega_1 \sin(\omega_1 t + \Phi)\}\end{aligned}$$

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I now look at the harmonic oscillator in the presence of damping, but weak damping. We had found a solution of the following form. We had an e power *minus gamma t* that multiplied a cosine. We had omega 1 slightly different from omega 0. We can have a phase shift. From this x , I calculate \dot{x} to calculate the kinetic energy. So, if I derive, there I have the product of two terms, I will end up with two terms. One term when I derive e power *minus gamma t* I have the *minus gamma* which appears times the cosine. If I derive the cosine: *less omega times sine*. So there I have two terms. Now as is often the case in physics, we use these equations to express a special case. The special case we have is that the damping is small. I don't want to drag all these terms with me without first expressing the fact that I want to put myself in a physical situation where gamma is much smaller than omega. So I do it here. I have a gamma term here, I have an omega term here. Omega 1 is next to omega 0 so this term is much larger than that one. So to express this situation there, of the velocity, I'm only going to keep the term in *sinus omega t*.

Notes

Summary



3m 09s

Energie, avec amortissement faible

Oscillateur harmonique faiblement amorti :

$$x = e^{-\gamma t} C \cos(\omega_1 t + \Phi) \quad \gamma \ll \omega_0$$

$$\dot{x} = C e^{-\gamma t} \{-\gamma \cos(\omega_1 t + \Phi) - \omega_1 \sin(\omega_1 t + \Phi)\} \\ \cong C e^{-\gamma t} \{-\omega_1 \sin(\omega_1 t + \Phi)\}$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \cong \left[\frac{1}{2} m C^2 \omega_1^2 \sin^2(\omega_1 t + \Phi) + \frac{1}{2} k C^2 \cos^2(\omega_1 t + \Phi) \right] e^{-2\gamma t}$$

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0 = \sqrt{\frac{k}{m}}$$

$$E(t) = \frac{1}{2} k C^2 e^{-t/\tau} \quad \tau = \frac{1}{2\gamma}$$

Mécanique | 2013 20

When I calculate the total energy, I have the kinetic energy plus the potential energy. And when I do the math, I have the term we just found here in *C square omega 1 squared times the sin squared*. And then here we have *k times x squared*, I wrote the *x squared* here. This is simply my *x-squared* with the exponential part I took out. Now, I'll remind you of a formula we had worked out for omega 1. If gamma is much smaller than omega 1, omega 1 is roughly equal to omega 0 which is equal to root of k over m. So in the formula here, the omega 1 squared, I will replace it with k over m. At that point, I have *times m*. So all that's left is the *k times C squared*, that's the same coefficient we have here. And again, we end up with a sin square and a cos square. I clean this up. This is the result we get. For the energy which now depends on time. There is no contradiction, we said we had a damped harmonic oscillator. So now we have a frictional force. As I was saying earlier, in the other module. A frictional force is not a force that can be derived from a potential. So we don't have the energy constant exactly as we expect to find it.

Notes

Summary



4m 47s

Energie, avec amortissement faible

Oscillateur harmonique faiblement amorti :

$$x = e^{-\gamma t} C \cos(\omega_1 t + \Phi) \quad \gamma \ll \omega_0$$

$$\begin{aligned} \dot{x} &= C e^{-\gamma t} \{-\gamma \cos(\omega_1 t + \Phi) - \omega_1 \sin(\omega_1 t + \Phi)\} \\ &\cong C e^{-\gamma t} \{-\omega_1 \sin(\omega_1 t + \Phi)\} \end{aligned}$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \cong \left[\frac{1}{2} m C^2 \omega_1^2 \sin^2(\omega_1 t + \Phi) + \frac{1}{2} k C^2 \cos^2(\omega_1 t + \Phi) \right] e^{-2\gamma t}$$

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0 = \sqrt{\frac{k}{m}}$$

$$E(t) = \frac{1}{2} k C^2 e^{-t/\tau} \quad \tau = \frac{1}{2\gamma}$$

\uparrow
[2\gamma] = \frac{1}{\tau}

Mécanique | 2013 20

If we let a harmonic oscillator oscillate with friction, it will stop. Its energy will tend towards 0. That's what I got here. I defined tau as one over 2 gamma. The 2 gamma here multiplies this 2 gamma here, multiplies the time, so the units of 2 gamma, it must be a time and that was written as tau. This is the same tau we had in the previous lesson.

Notes

Summary



6m 30s

$$Q = 2\pi \frac{(\text{énergie emmagasinée dans l'oscillateur})}{\text{énergie dissipée dans un cycle}}$$

énergie emmagasinée dans l'oscillateur = E

$$E(t) = \frac{1}{2} k C^2 e^{-t/\tau}$$

$$\text{énergie dissipée dans un cycle} = \frac{2\pi}{\omega_1} \left| \frac{dE}{dt} \right| = \frac{2\pi}{\omega_1} \frac{1}{\tau} E$$

$$Q = \omega_1 \tau$$

Now I can define the quality factor. We define in the technique a quality factor with this formula: 2π times the energy stored in the oscillator divided by the energy dissipated in one cycle, during one cycle if you like. So I remind you, the energy is the E I just found. I remind you what E is. To calculate the energy dissipated in a cycle, we'll say that it's worth this term: what is it? It's the period times dE over dt . You have a loss of energy per unit time here, times the time; that's during a cycle so you multiply by the period to get the energy dissipated in a cycle. Now, if this is energy, dE over dt that's 1 over τ times E . So dE over dt is 1 over τ times E . I have the 2π over ω_1 for the period. If now I put this dissipated energy here, the E energy there, the E 's simplify, it remains that the Q defined here is worth ω_1 times τ . This is a result that is often used in the technical world, You notice that it depends on τ . τ expressed the time over which the amplitude was decreasing the longer we have an oscillator that oscillates freely the greater its quality factor Q .

Notes

Summary



$$Q = 2\pi \frac{(\text{énergie emmagasinée dans l'oscillateur})}{\text{énergie dissipée dans un cycle}}$$

$$\text{énergie emmagasinée dans l'oscillateur} = E$$

$$E(t) = \frac{1}{2} k C^2 e^{-t/\tau}$$

$$\text{énergie dissipée dans un cycle} = \frac{2\pi}{\omega_1} \left| \frac{dE}{dt} \right| = \frac{2\pi}{\omega_1} \frac{1}{\tau} E$$

$$Q = \omega_1 \tau$$

$$\text{Amplitude à la résonance} \quad \frac{\rho(\omega = \omega_0)}{\rho(\omega = 0)} = \omega_0 \tau \cong Q$$

We can take up the question of the amplitude at resonance. Here is rho related to the zero frequency, we had found that it was worth omega 0 times tau Omega 0 is not very different from omega 1 so here what we just found is Q. We understand why people call Q the quality factor, because it tells us how much the oscillation will be amplified at the resonance frequency compared to what we would have at zero frequency. So it is indeed something that tells us the quality of a resonator.

Notes

Summary

