ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Potentiel et énergie potentielle

Mécanique, cours 11.1

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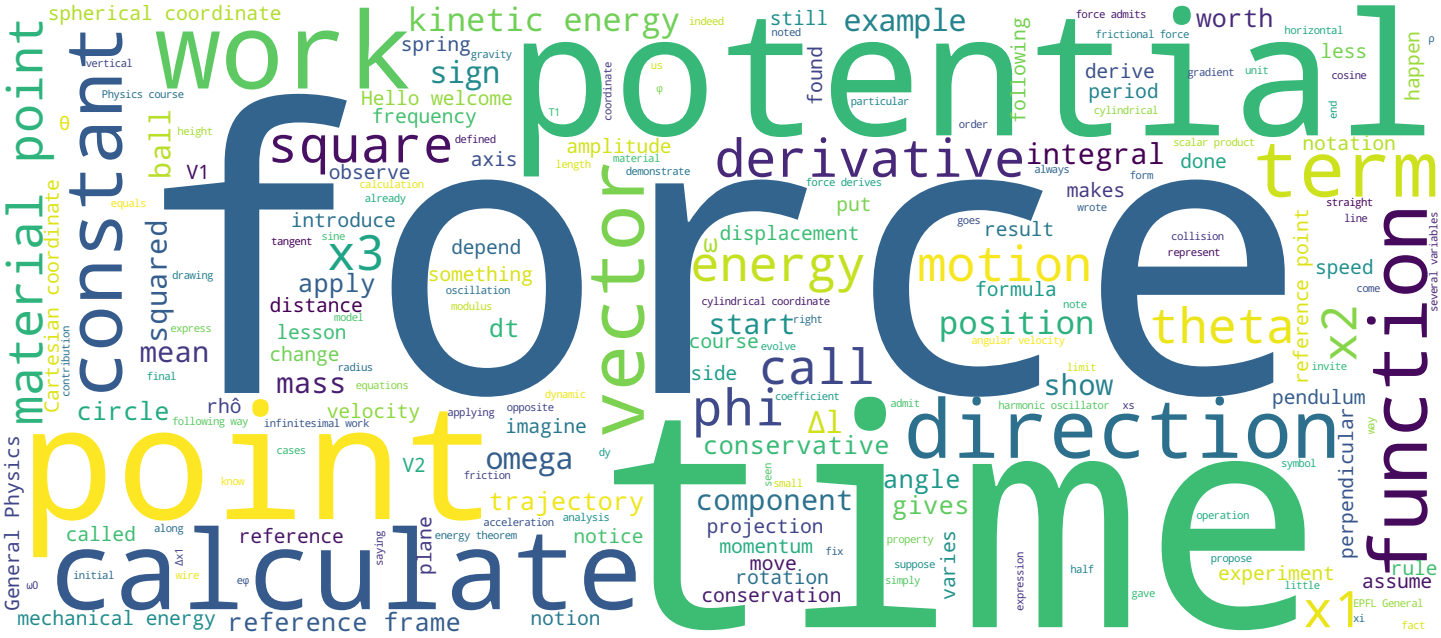
http://go.epfl.ch/traite-meca-1-11



Video

Potentiel et énergie potentielle

- Potentiel
- Energie mécanique
- Force dérivant d'un potentiel
- Exemples

Hello, welcome to the EPFL General Physics course. In this lesson, I would like to continue discussing the notion of energy. Until now, we have seen the notion of work and kinetic energy, and the kinetic energy theorem told us : if the kinetic energy has changed, then work has been done. I would now like to complete this vision with the introduction of the notion of potential of a force. Then, we will see in which cases a force admits a potential, we will be able in these cases to define what we call the mechanical energy, or the total energy, and we will see that, in these systems, the energy is a constant of the motion, which will be very useful for the analysis of the dynamics of material system. I will define what we mean when we say that a force derives from a potential, and I will show some examples where we derive the force given the potential, or the opposite, given the force, we find the potential.

Notes

Summary

0m 04s

Définition : potentiel d'une force

On considère une force qui dépend de la position et on suppose qu'on peut écrire :

$$W_{12} = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2)$$

(choix du signe justifié plus loin)

Position de référence (convention) : $\boldsymbol{r}_s$

$$V(\boldsymbol{r}) = \int_{\boldsymbol{r}}^{\boldsymbol{r}_s} \boldsymbol{F} \cdot d\boldsymbol{r} = V(\boldsymbol{r}) - V(\boldsymbol{r}_s)$$

$= 0$

I start with the definition of the potential of a force. I want to consider a force that depends on position. And I imagine that it is possible to make the following definition. I imagine that there is a function *V*, like this, such that the work of the force to get from point 1 to point 2 can be calculated simply by knowing the value of this function in ***r1*** and in ***r2***. When we calculate the work of the force, we go along the trajectory, and here we have values that depend only on 2 positions of the trajectory. So this is something special. Now, you'll notice that in this definition, the 1 comes here, the 2 here. This choice of sign will soon be justified when we discuss conservation of energy. In practice, it is convenient to use a reference point, and we calculate the potential as the work to get from the point under consideration, here we go from the point ***r***, to a point ***r****s*, called the reference point. You'll notice that if I apply that definition to this formula, here, I'll find *V* of ***r*** minus *V* of ***r****s*. Then, since we have *V* of ***r*** on both sides, this is equivalent to saying *V* of ***r****s* is zero.

Notes

Summary

1m 22s

Définition : potentiel d'une force

On considère une force qui dépend de la position et on suppose qu'on peut écrire :

$$W_{12} = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2)$$

(choix du signe justifié plus loin)

Position de référence (convention) : $\boldsymbol{r}_s$

$$V(\boldsymbol{r}) = \int_{\boldsymbol{r}}^{\boldsymbol{r}_s} \boldsymbol{F} \cdot d\boldsymbol{r}$$: ***potentiel de la force***

Cette définition n'a un sens que si le travail ne dépend pas du cher mais seulement du point de départ et du point d'arrivée.

On dit que la force est ***conservative***.

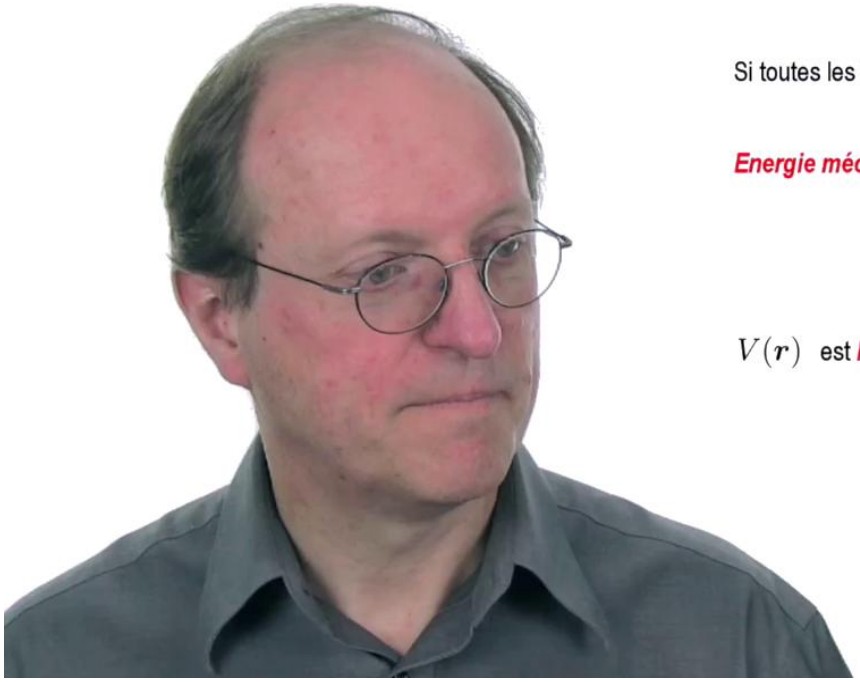
So to give oneself this point ***r***s, is to say where the potential is zero, is to give the potential zero. This *V* function of ***r*** is called the potential of the force. So we had a particular situation, and we can see that, if you have something like a frictional force, in particular, you can't have a potential. When you have a frictional force, the longer the trajectory, the more work the force will do. It will be impossible to define a function which gives the work simply as a function of the 2 positions, initial and final. We will see later some examples of forces which admit a potential. When a force admits a potential, we say that this force is conservative.

Notes

Summary

3m 18s

Définition : énergie mécanique



Si toutes les forces sont conservatives :

Energie mécanique : $E = T + V$

$V(\boldsymbol{r})$ est ***l'énergie potentielle***.

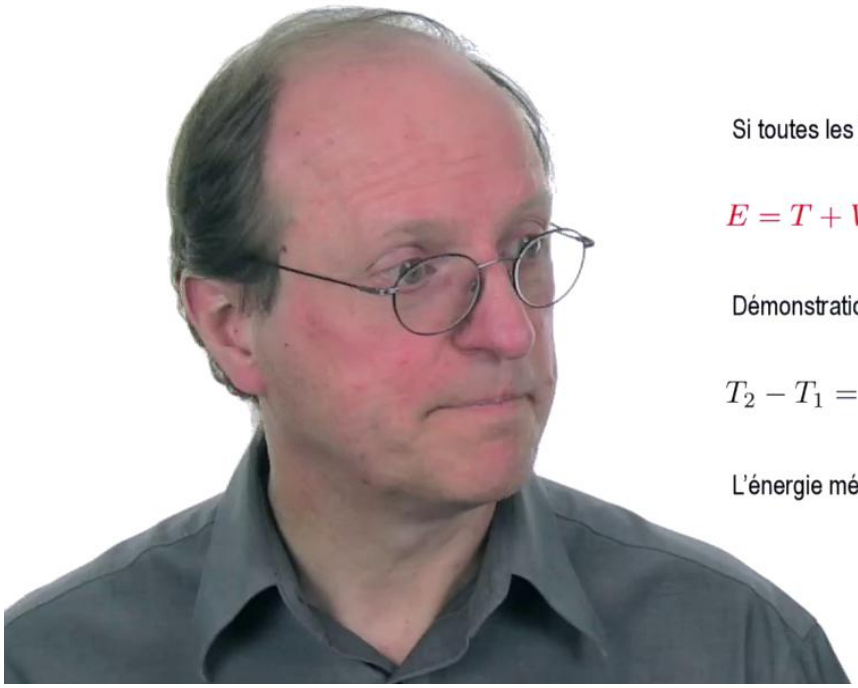
Now, mechanical energy, if all forces are conservative, we'll call mechanical energy *T* + *V*, where *V* is all the contributions of the potentials of all forces. And it is customary to call *V* the potential energy.

Notes

Summary

4m 22s

Propriété : énergie, constante du mouvement



Si toutes les forces sont conservatives :

$$E = T + V = \text{constante}$$

Démonstration : théorème de l'énergie cinétique,

$$T_2 - T_1 = W_{12} = V_1 - V_2 \implies T_2 + V_2 = T_1 + V_1$$

L'énergie mécanique est une ***constante du mouvement***.

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Now, I show an essential property, that energy is a constant of motion, when all forces are conservative. So I will say: energy is a constant. I demonstrate it in the following way. You look at the variation of the kinetic energy, it's worth the work, that's the kinetic energy theorem that gives us. And now, if all forces are conservative, I can calculate this work as *V1* - *V2*, we always have 1 to 2 in that order. And now you understand why we gave that order in the definition of *W1 2*, or if you want in the definition of *V*. We're going to the reference, so that now I can pass my *V2* on that side, pass the *T1* on the other side of the equal sign. So I have *T2* + *V2* which is *T1* + *V1*, that means *T* + *V* is constant. The mechanical energy of a system where all forces are conservative is a constant of motion, this is an extremely useful property that we can use to analyze the dynamics of a material system.

Notes

Summary

4m 45s

Propriété : la force dérive de son potentiel

Repère lié au référentiel : $(O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ $V(x_1, x_2, x_3)$

Si le potentiel existe : $\boldsymbol{F} = \sum_i -\frac{\partial V}{\partial x_i} \boldsymbol{e}_i$

$$\frac{\partial V}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{V(x_1 + \Delta x_1, x_2, x_3) - V(x_1, x_2, x_3)}{\Delta x_1}$$

Now I explain the idea that a force derives from its potential. You suppose that we have attached a reference frame to our frame of reference, and we will locate the position of a material point by its cartesian coordinates. If we work in Cartesian coordinates, we can calculate the force from the potential *V*, by doing these derivatives there. Here is a notation that is probably new to you. What this derivative means, here we have a derivative of *V* with respect to *xi*, where *i* is 1, 2 or 3. So we take that function there, and derive it with respect to *x1* or *x2* or *x3* leaving the other coordinates fixed, that's what this notation means, there's nothing more to find in there. If you like, I can write *d* of *V*, for example on *d* of *x1*, this is the limit when *Δx1* tends to zero of *V* of (*x1 + Δx1*,*x2*,*x3*) - *V* (*x1*,*x2*,*x3*) on *Δx1*. So this is the definition of a derivative where you notice that here I have the same *x2*, and there I have the same *x3*. So all I've done, when I say *∂V* over *∂x1* with *∂* (d round) like that, I'm dealing with a function of several variables, and I say: there's only the variable *x1* that varies, I fix *x2*, I fix *x3*.

Notes

Summary

6m 22s

Propriété : la force dérive de son potentiel

Repère lié au référentiel : $(O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ $\quad V(x_1, x_2, x_3)$

Si le potentiel existe : $\boldsymbol{F} = \sum_i -\frac{\partial V}{\partial x_i} \boldsymbol{e}_i$

Notation : $\boldsymbol{F} = -\boldsymbol{\nabla} V$ — NABLA

↳ gradient de V

So, that's the result I announce, and now I still have to prove it, but I give you a notation, I introduce here a symbol that we call a NABLA, it's the symbol that represents that operation; and we call that operation calculate the gradient of *V*. I taught you to calculate the gradient of *V* in the case where *V* is expressed in Cartesian coordinates. If we work in cylindrical or spherical coordinates, things go differently, but we won't need it in the context of this course, it's something I leave to other teachers.

Notes

Summary

8m 19s

Propriété : la force dérive de son potentiel

Repère lié au référentiel : $(O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ $\quad V(x_1, x_2, x_3)$

Si le potentiel existe : $\boldsymbol{F} = \sum_i -\frac{\partial V}{\partial x_i}\boldsymbol{e}_i$ $\quad$ Notation : $\boldsymbol{F} = -\boldsymbol{\nabla} V$

Travail dans un déplacement infinitésimal :

$$\boldsymbol{F} \cdot d\boldsymbol{r} = V(\boldsymbol{r}) - V(\boldsymbol{r} + d\boldsymbol{r})$$

$$d\boldsymbol{r} = \Delta\ell \boldsymbol{e}_i$$

$$\boldsymbol{F} \cdot d\boldsymbol{r} = \Delta\ell\, \boldsymbol{F} \cdot \boldsymbol{e}_i = F_i\, \Delta\ell = -\left[V(\boldsymbol{r} + \Delta\ell\, \boldsymbol{e}_i) - V(\boldsymbol{r})\right] = -\frac{\partial V}{\partial x_i}\Delta\ell$$

$i=1$ $\quad V(x_1 + \Delta\ell, x_2, x_3) - V(x_1, x_2, x_3)$

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Now, I still have to demonstrate this property I announced here, and I'll do it in the following way. Let us consider an infinitesimal work, and we apply the definition of *V*, so this infinitesimal work must be *V* of ***r*** initial, it is ***r***, and the ***r*** final is ***r*** + ***dr***. That was my *V1* - *V2*. Now, I take a special case. I make a *Δl* shift in the ***ei*** direction. And I make explicit the ***F***.***dr***, so I have the *Δl* times ***F*** times ***ei***, and there, of course, I recognize the *i* component of ***F***. Now, I'll work that expression into ***V*** with the - (minus) highlighted, so I'll have ***V*** of ***r*** +***dr*** minus ***V*** of ***r***, with the ***dr*** being worth *Δl* times ***e***. *And here I'm applying the rule of limited expansion, because if I take the component, if I take i =1, for example, in the parenthesis, here, I have* ***V*** *of (x1 + Δl, x2, x3) minus* ***V*** *of (x1, x2, x3).*

Notes

Summary

9m 09s

Propriété : la force dérive de son potentiel

Repère lié au référentiel : $(O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ $\quad V(x_1, x_2, x_3)$

Si le potentiel existe : $\boldsymbol{F} = \sum_i -\frac{\partial V}{\partial x_i}\boldsymbol{e}_i$ $\quad$ Notation : $\boldsymbol{F} = -\boldsymbol{\nabla} V$

Travail dans un déplacement infinitésimal :

$$\boldsymbol{F} \cdot d\boldsymbol{r} = V(\boldsymbol{r}) - V(\boldsymbol{r} + d\boldsymbol{r})$$

$$d\boldsymbol{r} = \Delta\ell \boldsymbol{e}_i$$

$$\boldsymbol{F} \cdot d\boldsymbol{r} = \Delta\ell\, \boldsymbol{F} \cdot \boldsymbol{e}_i = F_i\, \Delta\ell = -[V(\boldsymbol{r} + \Delta\ell\, \boldsymbol{e}_i) - V(\boldsymbol{r})] = -\frac{\partial V}{\partial x_i}\Delta\ell$$

$i = 1$ $\quad V(x_1 + \Delta\ell, x_2, x_3) - V(x_1, x_2, x_3)$

Mécanique | 2013 31

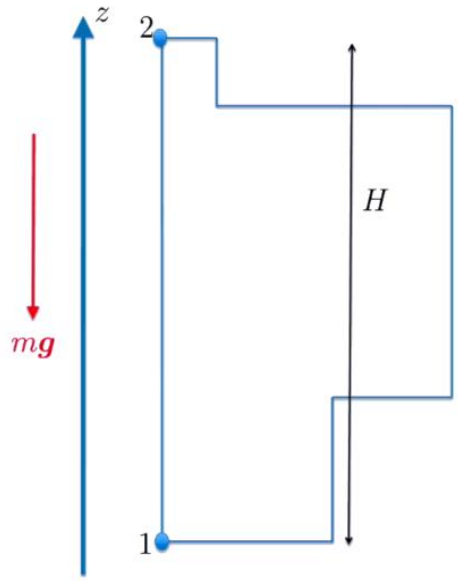
*And I note that x2 and x3 are like constants in this operation, and here I have like a function of x1 only, and I can apply my rule of limited development, with the derivative, the derivative, here, is given in ∂ because **V** is a function of several variables. And so I have times the Δl, the Δl that appears there. The minus sign, we already had that before. And now I see **F**i here, times Δl, and I have minus ∂**V** on ∂xi times Δl. So I found that for the i component of **F**, I have less ∂**V** on ∂xi, which is what I noted here. So I demonstrated this property. This is called deriving the force from its potential.*

Notes

Summary

10m 56s

Exemple : potentiel de la pesanteur



$$\boldsymbol{r}_s = \boldsymbol{r}_1$$

$$V(\boldsymbol{r}_2) = \int_{\boldsymbol{r}_2}^{\boldsymbol{r}_1} m\boldsymbol{g}d\boldsymbol{r}$$

$$= m\boldsymbol{g}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = mgH$$

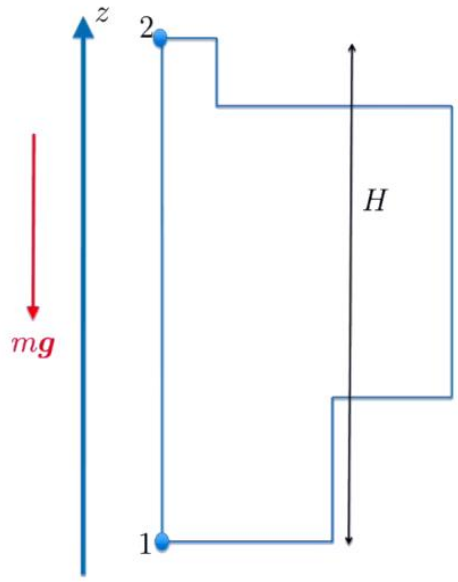
*I suggest you look at the potential of gravity. This is what symbolizes gravity for a material point of mass m. I choose an upward facing axis system. Here are my two points, 1 and 2. Let's say I agree that the potential reference point is at **r**1. That means I admit a zero of the gravity potential at position 1. The potential at position 2, is the integral that goes from 2 to 1 (that 1 there, because that's my reference) of the force times the displacement d**r**. Now, I can go straight from 2 to 1, but I can zigzag, and when I calculate a m**g**d**r**, if I have a d**r**, here, that is horizontal, that d**r** is perpendicular to m**g**, there we have a scalar product that gives zero, so we'll have no contribution from all those elements there, there are only the vertical parts that contribute, so I may as well calculate that straight line. I'll find m**g** times (**r**1 - **r**2), the **r**1 - **r**2 vector, it's there, it's in the direction of m**g**, so we end up finding m**g** times the height of point 2 above point 1, that's what I noted here.*

Notes

Summary

11m 56s

Exemple : potentiel de la pesanteur



$$\boldsymbol{r}_s = \boldsymbol{r}_1$$

$$V(\boldsymbol{r}_2) = \int_{\boldsymbol{r}_2}^{\boldsymbol{r}_1} m\boldsymbol{g}d\boldsymbol{r}$$

$$= m\boldsymbol{g}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = mgH$$

$$V(z) = mgz$$

$$F_z = -\frac{dV}{dz} = -mg$$

*Now we can do the reverse work, starting from the potential. I will write that **V** at height z above the zero of my potential, this is m**gz**, and I can calculate the force in direction **z**. Here I have a straight d because **V** is a function of **z** only. So I have less d**V** on d**z**, applying the rule I just gave to find the force from the potential. And the derivative is m**g**, with the minus sign, that's right: the projection of m**g** on that axis, which is up, has a minus m**g**.*

Notes

Summary

13m 39s

Exemple : potentiel de la force d'un ressort

Force:

$$F(x) = -kx$$

$$x_s = 0$$

$$V(x) = \int_x^{x_s} F(y)\,dy = \int_0^x ky\,dy = \frac{1}{2}kx^2$$

As a second example, I take the potential force of a spring. Let's express the force of the spring like this. Let's say the rest position of the spring defines the zero of the potential. I now apply my definition of the potential ***V****. So I write* ***V*** *as the work to get from x to xs. And the calculation, the integral, I write it here with* ***F*** *of ydy, so that we don't confuse that x there with the function of the position. xs is zero, so I do an integral from zero to x changing sign. I have the force* ***F*** *of y worth less k times y, the 2 signs cancel out. There remains kydy, and that, the integral of ydy, makes y square out of 2, to be evaluated at x and zero, makes 1/2 of kx square. It happens very often, unfortunately, that students say to themselves: force is k x, displacement is x, so work is kx square. That is a false reasoning, the force, as x evolves, and there I purposely wrote y, y evolves from zero to x, and each time, the force changes, so it is indeed this integral that we must calculate to calculate the work.*

Notes

Summary

14m 23s

Exemple : potentiel de la force d'un ressort

Force:

$$F(x) = -kx$$

$$x_s = 0$$

$$V(x) = \int_x^{x_s} F(y)\,dy \;\; = \int_0^x ky\,dy = \frac{1}{2}kx^2$$

$$F_x = -\frac{dV}{dx} = -kx$$

*Now we can work in the other direction, given the **V** of x, we can calculate the force in the x direction by calculating the gradient, which is simply minus d**V** on dx, and we find well minus k x, which is correct, which is what we had given in the beginning.*

Notes

Summary

16m 05s