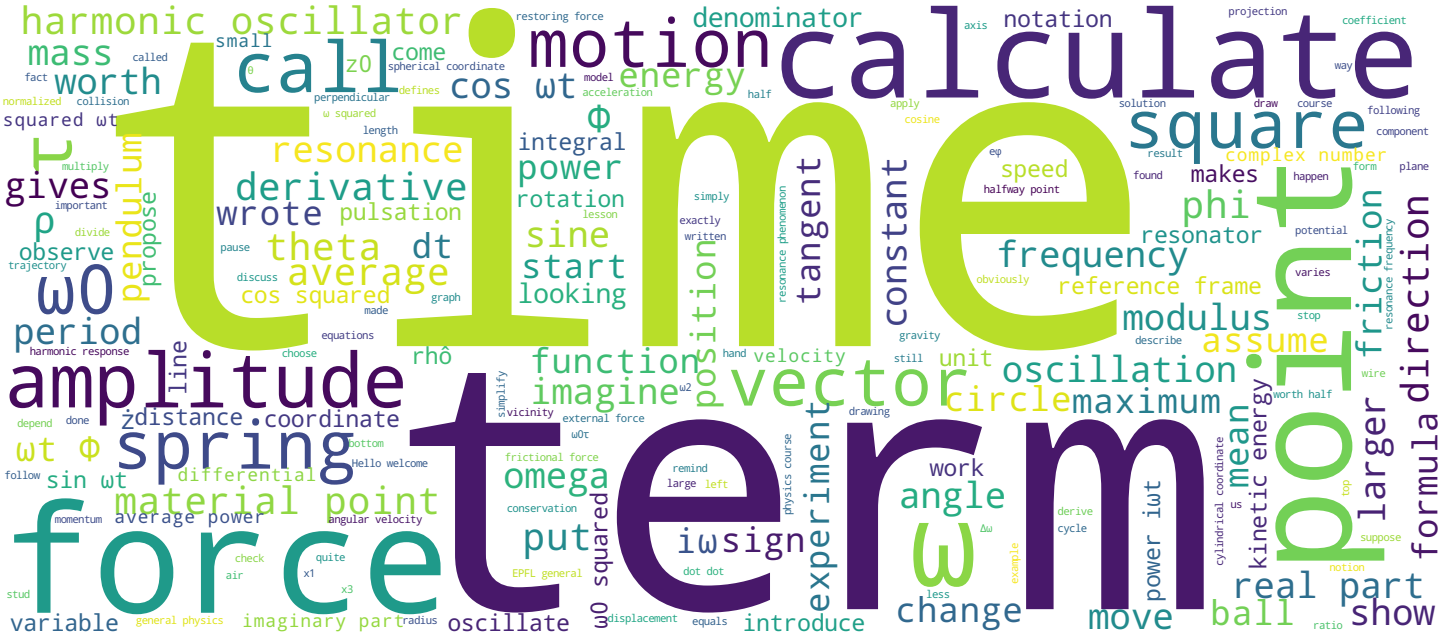
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Le phénomène de résonance

Mécanique, cours 10.2

Jean-Philippe Ansermet

http://go.epfl.ch/traite-meca-2-10



Video

EPFL

Le phénomène de résonance

- Equation du mouvement
- Réponse harmonique
- Spectre
- Amplitude à la résonance
- Puissance

Mécanique | 2013 7

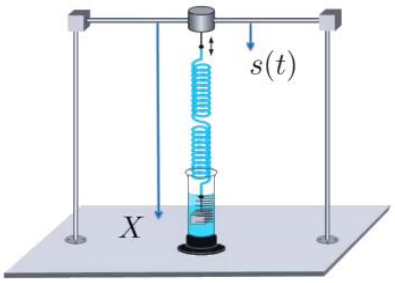
Hello, welcome to the EPFL general physics course. In this lesson I'm going to introduce a very important concept, the resonance phenomenon, I'm doing it here because we're going to apply a new tool, the notion of power and we'll see that in this problem we get a good description of what happens when we examine the power of a force. The phenomenon of resonance is very important for engineers those for example who have to build structures and are concerned about avoiding having resonance natural frequencies, as we will see, which are in the vicinity of the frequencies of the use that we make of these structures; or you have scientists or engineers who have to develop ultra-sensitive sensors and we will see why we often resort to resonators in order to obtain extremely sensitive measurements. I will first pose the problem as an equation of motion for a harmonic oscillator with an applied force, I will look for the harmonic response, in the experiment videos I show practically what the harmonic response means, we will see it here in mathematical form, I will discuss the spectrum, i.e. what amplitude we get for what frequency, we will see a particular property, namely the sensitivity of the resonators when we excite them at the resonance frequency and finally, to apply the previous lesson, we are going to calculate the power of the force that solicits our resonator.

Notes

Summary

0m 04s

Equation du mouvement



$$m\ddot{X} = -k\left(X - s(t) - \ell\right) - b\dot{X} + mg$$

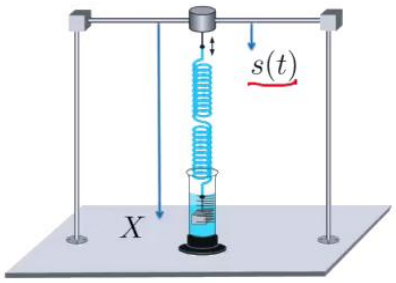
I imagine the following situation: you have a spring, a mass attached to the spring, we'll suppose that we can't neglect the friction but that the friction is low so we have a lot of oscillations before we have a significant damping. We suppose here that our system of spring and mass, here is the spring and the mass, is suspended at this point to a vibrating pot, an electromagnetic system, which shakes this point of attachment; to describe the kinematics of the system I only have to give myself an X coordinate which defines the position of the stud and then s which depends on time, which is an experimental data, and which defines the position of the attachment point; to write the equation of motion there is no great difficulty I can skip the steps and immediately present it like this I applied *f=ma* the acceleration is simply the second derivative of the position I have a restoring force what I need to indicate here is the force that is exerted at this point on the spring stud and so it is a restoring force, hence the minus sign, a force proportional to the elongation because if the coefficient of proportionality and the elongation of the spring it's *X - s X - s* is that distance minus the rest length of the spring; I've also included the assumed frictional force for the liquid, and gravity.

Notes

Summary

2m 04s

Equation du mouvement



$$m\ddot{X} = -k\,(X - s(t) - \ell) - b\dot{X} + mg$$

$$F(t) = ks(t)$$

$$x = X - \ell - mg/k$$

$$m\ddot{x} = -k\,x - b\dot{x} + F(t)$$

$$F(t) = f\cos(\omega t)$$

donné

Now what I would like to do is to show that this problem is equivalent to a harmonic oscillator problem with a spring attached to a fixed point and an oscillating force applied on the material point; to do this I will make two substitutions, first I will call the term *KsF(t)* suggesting a force then I will change the variable I will consider small *x* which is large *X* minus *l* (the length at rest) and then I'll also include, as was done once before, the effect of gravity in that definition there and with that change of variable there of course the derivatives of *x* are equal to the derivatives of *X* but my equation simplifies to take that look; now we recognize in this equation the one we would have for a harmonic oscillator defined by the variable *x*, a restoring force - *kx*, a frictional force and an external force. In what follows we will consider that this external force has the form, an amplitude *times cos (ωt)*, this *ω* there, attention, this *ω* there is given by the experiment and basically it comes from *s(t)*.

Notes

Summary

4m 01s

Passage aux nombres complexes

$$m\ddot{x} = -k\,x - b\dot{x} + F(t) \qquad F(t) = f\cos(\omega t)$$

On imagine un 'autre' problème :

$$+\,i \quad m\ddot{y} = -k\,y - b\dot{y} + F_y(t) \qquad F_y(t) = f\sin(\omega t)$$

$z = x + iy$

$\cos\omega t + i\sin\omega t = e^{i\omega t}$

There, now we'll try to integrate this differential equation with this *F* and to do this I propose to extend the problem by considering another problem which is mathematically extremely similar which I would write for a variable *y* where I would have essentially the same equation of motion but I imagine the force would not be a *cos(ωt)* but a *sin (ωt)* why am I doing this? Because you will see after an extremely compact form that I can write with complex numbers, indeed I imagine to make an addition of these two equations there, I ignore the text bit, I multiply by *i* here and add and I'll put $z = x + iy$ then the derivatives are immediate $\dot{z}$ will give $\dot{x}$ plus $i\dot{y}$, etc. so we see appearing here a $\ddot{x} + i\ddot{y}$ that's going to give us *z dot dot* here we have $x + y$ that's going to give *z* here $\dot{x} + i\dot{y}$ that's going to give $\dot{z}$ and here we have, perhaps it should be remembered, we have a $\cos\omega t + i \sin\omega t$ and that I hope you've already seen or I'll teach you it's worth "*e* power *ωt*"; with that we arrive at this equation of motion in complex form.

Notes

Summary

5m 32s

Passage aux nombres complexes

$$m\ddot{x} = -k\,x - b\dot{x} + F(t) \qquad F(t) = f\cos(\omega t)$$

On imagine un 'autre' problème :

$$m\ddot{y} = -k\,y - b\dot{y} + F_y(t) \qquad F_y(t) = f\sin(\omega t)$$

On multiplie par (i) et on somme :

$$m\ddot{z} = -k\,z - b\dot{z} + f\,e^{i\omega t}$$

Paramètres avec notation explicite :

$$\ddot{z} + \frac{1}{\tau}\dot{z} + \omega_0^2 z = \alpha\,e^{i\omega t}$$

Hypothèse du régime harmonique :

$$z = z_0\,e^{i\omega t}$$

Now from the complex form how do we find our original problem? That's easy, just take the real part. To simplify the writings and as usual with notations that are explicit, I will write that the ratio *k* over *m* I will write it "*ω0 squared*" because if I had a harmonic oscillator with just these two terms and without the other two I would have a proper pulsation, as they say, the pulsation we would get would be the *ω0* = √*k over m* so I choose to call *k on m ω0*; if I divide by *m* I have a *b* on *m* which multiplies *ż* which must have the same unit as *z dot dot*, so *b on m* has the unit of *1 on a time* here I choose the notation *1 on τ* and finally *f on m* I decided to call it *α*. Now I make a particular assumption: I am not looking for the general solution to this differential equation, I am looking for a so-called harmonic solution, that means I assume that the transients that can be observed in the experiment videos these transients have already damped, I'm waiting long enough to only have an oscillation at the same pulsation as the pulsation of the applied force, so I'm looking for a solution for this differential equation that oscillates with the pulsation *ω*.

Notes

Summary

7m 21s

Passage aux nombres complexes

$$m\ddot{x} = -k\,x - b\dot{x} + F(t) \qquad F(t) = f\cos(\omega t)$$

On imagine un 'autre' problème :

$$m\ddot{y} = -k\,y - b\dot{y} + F_y(t) \qquad F_y(t) = f\sin(\omega t)$$

On multiplie par (i) et on somme :

$$m\ddot{z} = -k\,z - b\dot{z} + f\,e^{i\omega t}$$

Paramètres avec notation explicite :

$$\ddot{z} + \frac{1}{\tau}\dot{z} + \omega_0^2 z = \alpha\,e^{i\omega t}$$

Hypothèse du régime harmonique :

$$z = z_0\,e^{i\omega t}$$

$$-\omega^2 z_0 + \frac{1}{\tau}(i\omega)z_0 + \omega_0^2 z_0 = \alpha \qquad z_0 = \frac{\alpha_0}{\left(-\omega^2 + (i\omega/\tau) + \omega_0^2\right)}$$

So if we put that *z* in that equation there when we derive that with respect to time we get *iω* that comes before the *z* when we put a *z dot dot* we get the square of *iω* which makes a - *ω2* here's the - *ω2* here's the *iω* here's the "we didn't derive" term and I can divide by the "e power iωt" everywhere, I have this equation where we have *z0* here we can solve for *z0* and this is what we find here's the *-ω2* the *ω0 square* that's there and the *iω on τ* that's there here's the amplitude here we have the amplitude of our complex problem oscillation.

Notes

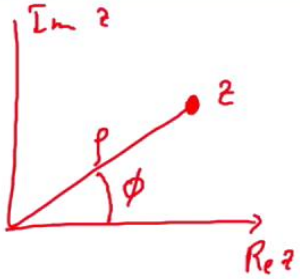
Summary

9m 13s

Réponse harmonique

$$z_0 = \frac{\alpha_0}{\left(-\omega^2 + (i\omega/\tau) + \omega_0^2\right)}$$

$$z_0 = \rho\, e^{i\phi}$$



I remind you the solution we just found here this is what we call the harmonic response now you will see that it is favorable to use this notation there so as a reminder if I draw the complex plane with *Real of z* here *Imaginary of z* here here's my complex number *z* the ρ I wrote down here is the modulus of *z* and the Φ is this angle here.

Notes

Summary

10m 03s

Réponse harmonique

$$z_0 = \frac{\alpha_0}{\left(-\omega^2 + (i\omega/\tau) + \omega_0^2\right)} \qquad z_0 = \rho\, e^{i\phi}$$

$$x = Re(z) = \rho\cos(\omega t + \phi)$$

$z = \rho e^{i\phi} e^{i\omega t} \Rightarrow Re(z) = \rho\cos(\omega t + \phi)$

So the modulus, first I can write *x* if we want the actual coordinate then it's simple because we have *z* which is *z0* so *ρ e power iΦ times e power iωt* and the real part of *z* it's going to give *ρ cos (ωt + Φ)* that's what I've written here; here, now if we want to calculate the modulus of *ρ*, that term there, well, we have to take the modulus of the numerator, that's *α0*, that's a real, then in the denominator we take the real part which contains those two terms there the real part squared plus the imaginary part squared root of all that, that gives me the *ρ*.

Notes

Summary

10m 43s

Réponse harmonique

$$z_0 = \frac{\alpha_0}{(-\omega^2 + (i\omega/\tau) + \omega_0^2)} \qquad z_0 = \rho\, e^{i\phi}$$

$$x = Re(z) = \rho\cos(\omega t + \phi) \qquad tg(\phi) = \frac{Im(z_0)}{Re(z_0)} = \frac{-\omega/\tau}{\omega_0^2 - \omega^2}$$

$$\rho = \frac{\alpha_0}{\sqrt{\left((\omega/\tau)^2 + \left(\omega_0^2 - \omega^2\right)^2\right)}}$$

Now we need to calculate the phase so we can calculate *tangent of Φ* we can do it like this so the *tangent of Φ* obviously if I redo my sketch here the real number *z* with the angle *Φ* here the tangent *Φ* is this distance divided by that one so it's the imaginary part divided by the real part, that's what I wrote here; now to find it we can multiply this complex number in the denominator by its complex conjugate so I put *-iω over τ + ω0 squared* I rewrite the same term at the top *-ω squared - iω over τ + ω0 squared* I write the same thing at the top and at the bottom at the bottom I have a number times its complex conjugate, so I have the modulus squared and now I'll be able to find the imaginary part it's proportional to this term *-ω over τ* that's what I wrote here and the real part with the same proportionality coefficient *ω0 squared - ω squared* that's what I wrote there.

Notes

Summary

11m 45s

Réponse harmonique

$$z_0 = \frac{\alpha_0}{\left(-\omega^2 + (i\omega/\tau) + \omega_0^2\right)} \qquad z_0 = \rho\, e^{i\phi}$$

$$x = Re(z) = \rho\cos(\omega t + \phi) \qquad tg(\phi) = \frac{Im(z_0)}{Re(z_0)} = \frac{-\omega/\tau}{\omega_0^2 - \omega^2}$$

$$\rho = \frac{\alpha_0}{\sqrt{\left((\omega/\tau)^2 + \left(\omega_0^2 - \omega^2\right)^2\right)}} \qquad \sin\phi = \frac{-\omega/\tau}{\sqrt{(\omega/\tau)^2 + (\omega_0^2 - \omega^2)^2}}$$

$$\frac{\rho(\omega = \omega_0)}{\rho(\omega = 0)} = \omega_0\tau \gg 1$$

We can also calculate the sine with the same considerations so the sine was the imaginary part divided by the modulus so we have that term there divided by the modulus which is here. I let you pause if you want to check this little complex number calculation. Now what's important from a physical point of view here it is I'm looking at the amplitude here if τ is very large the resonance is about $\omega 0$ later on I'm going to show you a graph where you'll see that it's not exactly $\omega 0$ that has the resonance frequency but the larger τ is, the more true it is and if you look at the amplitude ρ when ω is at the value $\omega 0$ and compare it to the amplitude ρ that you have when ω is zero it's a simple little algebraic calculation you'll find that you have $\omega 0$ *times* τ left. Now I remind you that we made the assumption that this number there $\omega 0\tau$ without unit, this number there is much larger than 1, so I'm saying that the amplitude, when the applied frequency is close to $\omega 0$, is much larger than the amplitude I have statically.

Notes

Summary

13m 13s

Réponse harmonique

$$z_0 = \frac{\alpha_0}{\left(-\omega^2 + (i\omega/\tau) + \omega_0^2\right)} \qquad z_0 = \rho\, e^{i\phi}$$

$$x = Re(z) = \rho\cos(\omega t + \phi) \qquad tg(\phi) = \frac{Im(z_0)}{Re(z_0)} = \frac{-\omega/\tau}{\omega_0^2 - \omega^2}$$

$$\rho = \frac{\alpha_0}{\sqrt{\left((\omega/\tau)^2 + \left(\omega_0^2 - \omega^2\right)^2\right)}} \qquad \sin\phi = \frac{-\omega/\tau}{\sqrt{(\omega/\tau)^2 + (\omega_0^2 - \omega^2)^2}}$$

$$\frac{\rho(\omega = \omega_0)}{\rho(\omega = 0)} = \omega_0\tau$$

Plus l'oscillateur libre s'amortit lentement, plus l'amplitude à la résonance est grande.

Think back to your mass hanging from a spring the ρ of $\omega = 0$ is the displacement of the mass you have that you go infinitely slowly in your *s* displacement of *t* and there you have the stud which moves by as much as *s* moves and you find that if now you make that movement at the frequency which is in the vicinity of the natural frequency of the harmonic oscillator you have a much, much greater effect. Now that's the first physical result we get on resonators and that's what I'm pointing out here the slower the free oscillator dampens the larger τ is the larger the amplitude at resonance here is a graph of this ratio ρ of ω normalized to the ρ we get at zero frequency as a function of the frequency normalized by this characteristic frequency $\omega 0$.

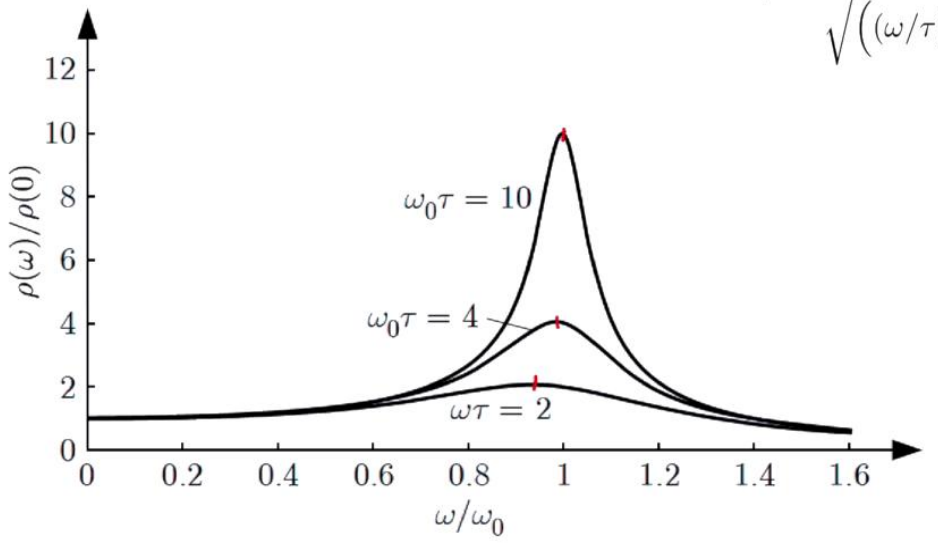
Notes

Summary

14m 46s

Spectre de la réponse harmonique

$$\rho = \frac{\alpha_0}{\sqrt{\left((\omega/\tau)^2 + \left(\omega_0^2 - \omega^2\right)^2\right)}}$$



So we see here what I'm going to call the amplitude spectrum and we have a maximum at resonance when *ω0τ* is 10 we have ten times more than we have at zero frequency; when *ω0τ* is 4 we have four times more, etc. and we detect here, we manage to distinguish more or less a small shift, which becomes less and less noticeable, of the frequency at maximum.

Notes

Summary

16m 03s

Puissance moyenne dissipée

Puissance instantanée : $P = \boldsymbol{F} \cdot \boldsymbol{v}$

Puissance moyenne :

Now since we've introduced the notion of power, we'll use it here you'll see that it gives a slightly different picture of our resonance phenomenon, I start with the instantaneous power *F times v* now I'd like to calculate the average power for a given ω, why? You imagine again or maybe you have to imagine that you hang this spring and mass on your finger and with your hand you exert a force on your harmonic oscillator; from time to time you will have the spring bounce against you, and from time to time the force will go the other way compared to the speed of the material point at the bottom of the spring so there will be a round trip of energy between the material point and your hand; what we'd like is the average over a cycle because what we'd like is to calculate over a cycle the energy we have to bring to the system to maintain this oscillation.

Notes

Summary

16m 40s

Puissance moyenne dissipée

Puissance instantanée : $P = \boldsymbol{F} \cdot \boldsymbol{v}$

Puissance moyenne : $\langle P \rangle = \langle \boldsymbol{F} \cdot \boldsymbol{v} \rangle = \langle f \cos(\omega t) Re(\dot{z}) \rangle$

$$\langle P \rangle = \frac{1}{T} \int_0^T P(t)\, dt$$

$$\dot{z} = i\omega\rho\, e^{i\phi}\, e^{i\omega t} = i\omega\rho\big(\cos(\omega t + \phi) + i \sin(\omega t + \phi)\big)$$

$$\langle P \rangle = \big\langle f \cos(\omega t) \sin(\omega t + \phi)(-\omega\rho) \big\rangle$$

We agree that it's a damped oscillator, if now I stop my excitation force the oscillator will stop so I'd like to describe the mechanical energy contribution that I give to this system to maintain the oscillation of constant amplitude in spite of the friction this is the calculation we want to do so I want to calculate what I will call the average power and here I introduce it's an angular parenthesis to mean the average, so I define the average as the integral of the power over a period between 0 and *t*, integral over time, I divide by the period so I do have an average power *f times v* that gives us the *f cos(ωt)* and the *v* that is here is, then as we calculated for *z* we have to take the real part of *ż ż* I can calculate it, it's *iω* times the *z0* times the *th power iωt, th power iωt*, I've said this before, it makes the cosine *ωt + Φ* plus *i sine ωt + Φ*. Now I want the real part so there I have an imaginary term because of the *i* that's there in front of it whereas here I have a *i square* that makes *-1* there I have a real term so it's the sine that gives me the real term, so my average power is therefore the average of *f cos(ωt) sin(ωt + Φ)* times this term, the minus comes from the *i square*, *ωρ* are here.

Notes

Summary

17m 56s

Puissance moyenne dissipée

Puissance instantanée : $P = \boldsymbol{F} \cdot \boldsymbol{v}$

Puissance moyenne : $\langle P \rangle = \langle \boldsymbol{F} \cdot \boldsymbol{v} \rangle = \langle f \cos(\omega t) Re(\dot{z}) \rangle$

$$\langle P \rangle = \frac{1}{T} \int_0^T P(t)\, dt$$

$$\dot{z} = i\omega\rho\, e^{i\phi}\, e^{i\omega t} = i\omega\rho\big(\cos(\omega t + \phi) + i \sin(\omega t + \phi)\big)$$

$$\begin{aligned}\langle P \rangle &= \langle f \cos(\omega t) \sin(\omega t + \phi)(-\omega\rho) \rangle \\ &= \langle (-\omega\rho f) \cos(\omega t)\big(\sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi)\big) \rangle = \langle (-\omega\rho) f \cos^2(\omega t) \sin(\phi) \rangle = \frac{-1}{2} f\omega\rho \sin(\phi)\end{aligned}$$

$$\langle P \rangle = \frac{1}{2} m\alpha_0^2 \tau \frac{1}{\left(\frac{\left(\omega_0^2 - \omega^2\right)^2}{(\omega/\tau)^2} + 1 \right)}$$

Now to calculate this average I want to get rid of this *ωt + Φ*, I do it in the following way: I expand the *sin(ωt + Φ)* into *sin cos*, *cos*, *sin* this is the elementary trigonometry rule; now here I have a *cos(ωt) times sin(ωt)*, which is worth half a sine of 2*ωt*, this term there over a period *t*, oscillates and on average is worth 0. Whereas the term here gives a *cos squared ωt* and *cos squared ωt* oscillates between 0 and 1 and its average is worth half. I'll let you pause if you want and check this very simple property the integral of *cos squared ωt* over one period. So I write here that the average of the *cos squared ωt* is worth half. I have the minus sign which I have taken up here, and then I have the sine *Φ*, the sine *Φ* we've already calculated it. And if I put its value I get this formula here.

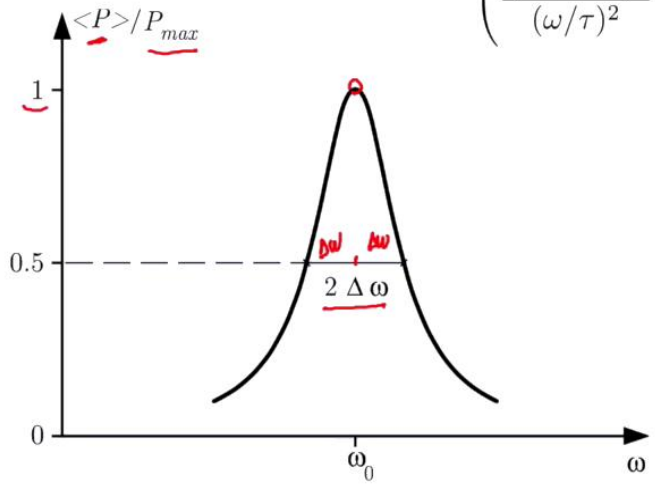
Notes

Summary

19m 54s

Spectre de puissance moyenne

$$\langle P \rangle = \frac{1}{2} m \alpha_0^2 \tau \frac{1}{\left(\frac{\left(\omega_0^2 - \omega^2\right)^2}{(\omega/\tau)^2} + 1 \right)}$$



I'm now going to show you a graph of that formula there so here's the *average P*. And what I propose to do for that purpose there is to normalize *P* with respect to the value at the maximum. You see I've normalized so that we have 1 at the maximum so this is a little analysis exercise you can do. You can calculate *Average P* as a function of *ω* and look for the extremum. It's not quite the same as before, we don't have exactly the same function, because if I multiply by *ω over τ squared* here we do have the same denominator but there we'll have a *ω squared* that appears, so it's not the same function anymore. If we look for its maximum we'll find that it's *ω0* whatever *τ* and when *ω* is worth *ω0 square* that term is zero all that denominator is worth 1 so we have that amplitude there and it's relative to that amplitude there, sorry, it's not an amplitude, it's relative to that power there that we normalized. And now in practice something that is extremely convenient is to discuss the width of that resonance so I have here, imagine I had *ω0*. I call this width *2 times Δω*, that means I have a *Δω* here, and there too; now when do we get to the halfway mark?

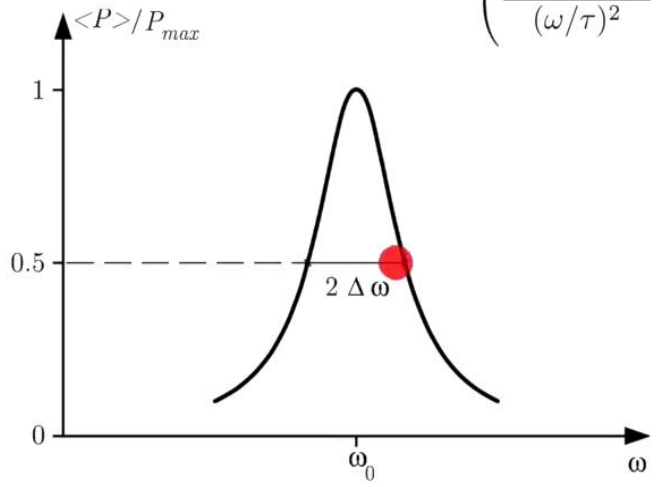
Notes

Summary

21m 16s

Spectre de puissance moyenne

$$\langle P \rangle = \frac{1}{2} m \alpha_0^2 \tau \frac{1}{\left(\frac{\left(\omega_0^2 - \omega^2\right)^2}{(\omega/\tau)^2} + 1 \right)}$$



Puissance moitié du maximum :

$$\frac{\left(\omega_0^2 - \omega^2\right)^2}{(\omega/\tau)^2} = 1$$

$$|\omega_0 - \omega|(\omega_0 + \omega) = (\omega/\tau)$$

$$\Delta\omega\, 2\omega_0 \cong (\omega_0/\tau)$$

$$2\Delta\omega = \frac{1}{\tau}$$

Here I have 1, here is the halfway point when do I get to the halfway point? It's worth saying that I have half of this It's when all of this is worth 1 because at that point I have a factor of 2 in the denominator, so I write that the condition for getting to the halfway point is this: this must be worth 1 which I could write as follows. I take the square root keeping the absolute value so here I have the *Δω* now I assume I have a narrow line. I'll make the approximation that this *ω* there is about *ω0* and this *ω* there is about *ω0* too to calculate the *Δω*. This gives me the following formula and if you simplify by *ω0* here is the practically important formula which tells us that the line width is 1 over the amplitude decay time when the harmonic oscillator is left free. So if you consider a harmonic oscillator, you shake it and leave it alone, you will observe the decay of the oscillation which is characterized by *τ*, the larger *τ* is, the narrower the resonance.

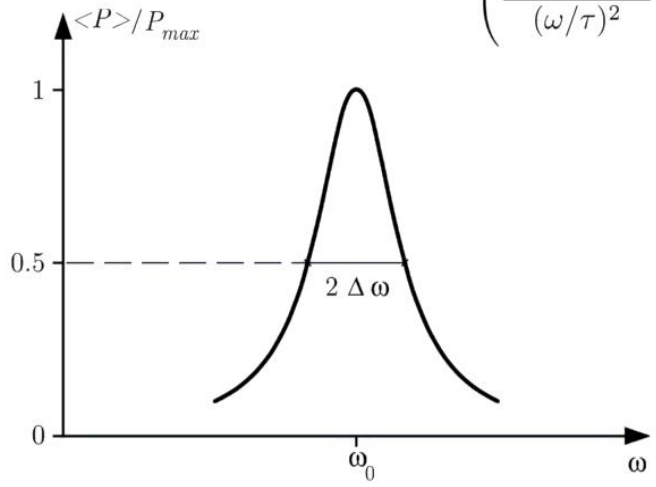
Notes

Summary

22m 59s

Spectre de puissance moyenne

$$\langle P \rangle = \frac{1}{2} m \alpha_0^2 \tau \frac{1}{\left(\frac{\left(\omega_0^2 - \omega^2\right)^2}{(\omega/\tau)^2} + 1 \right)}$$



Puissance moitié du maximum :

$$\frac{\left(\omega_0^2 - \omega^2\right)^2}{(\omega/\tau)^2} = 1$$

$$|\omega_0 - \omega|(\omega_0 + \omega) = (\omega/\tau)$$

$$\Delta\omega\, 2\omega_0 \cong (\omega_0/\tau)$$

$$2\Delta\omega = \frac{1}{\tau}$$

So when you take a pendulum a harmonic oscillator formed by a spring in air τ is very large, the line is very narrow, you just have to move a tiny bit relative to $\omega 0$ and the amplitude drops all at once; whereas when you are in the water τ is considerably shorter so the line is considerably wider and for the same frequency shift you still have quite a bit of amplitude.

Notes

Summary

24m 41s