

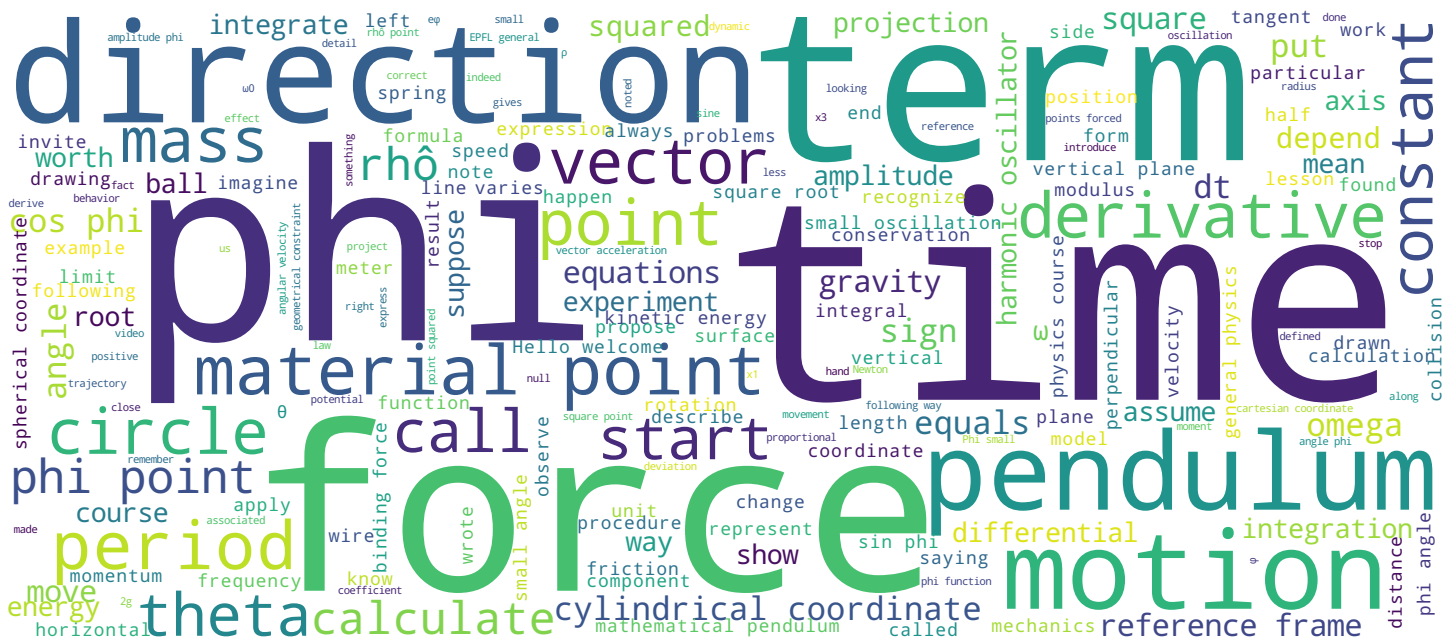


<http://go.epfl.ch/traite-meca-2-9>

Le pendule mathématique

Mécanique, cours 9.2

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EPFL

Video





- Contrainte géométrique
- Marche à suivre
- Petites oscillations
- Méthode d'intégration

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Hello, welcome to the EPFL general physics course. In this lesson, I deal with problems of material points forced to move on a given surface or line. Here, I will look in detail at one such problem, the problem of an idealized pendulum, called the mathematical pendulum. The pendulum problem has its historical importance, since Galileo, already, had observed the behavior of pendulums. In particular, he had noticed that the period of a pendulum does not depend on the mass that is attached to the pendulum. A result that we will find here. So, I will start by defining the pendulum as a material point problem with a geometrical constraint. I will then apply the procedure that I recommend for all mechanics problems. We'll get equations of motion, and we'll look at the behavior of these equations, or if you want, we'll look at the behavior predicted by these equations, when we have small oscillations of the pendulum around the equilibrium. Then, I will show how to approach the integration of this equation of motion, by a very useful method in mechanics.

Notes

Summary



0m 04s

Définition : le pendule mathématique plan



Point matériel :

- pesant (sous l'effet de la pesanteur)
- astreint à se déplacer sur un cercle
- dans un plan vertical
- sans frottement

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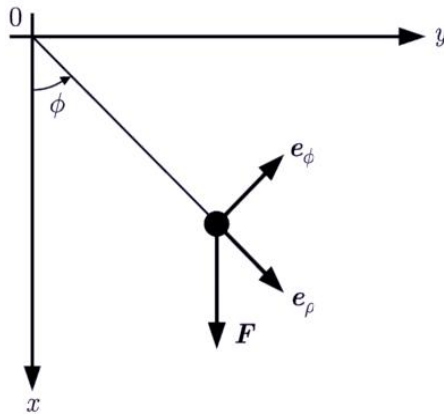
So, I start with the definition of what I will call the mathematical pendulum. First, we'll consider that we have a material point, we'll suppose that it's under the effect of gravity, we'll say that it's a weighing material point. We say that this material point is constrained to move on a circle, again, this is a way to evacuate all sorts of details about the mechanisms that ensure that the material point oscillates at a constant distance from a point of the reference frame. We suppose that the motion takes place in a vertical plane, and finally, that there is no friction.

Notes

Summary



1m 33s



Coordonnées cylindriques :

(ρ, ϕ, z)

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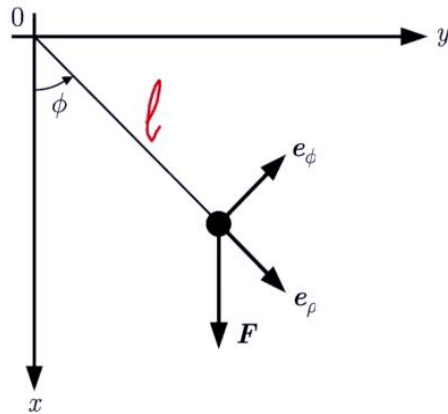
I start my procedure with the definition of the reference frame, for a pendulum problem, I just have to take the auditorium or the laboratory where I do my experiment. I materialize my frame of reference by a cartesian coordinate system. I represent the force of gravity, by noting it F , without specifying further what F is, except that it is vertical. I would like to describe the motion of this pendulum, with the coordinates that are the most convenient. Obviously, the motion of a pendulum is characterized by the oscillation or variation of this angle, so I'm going to propose to use cylindrical coordinates and use the ϕ angle of the cylindrical coordinates, to locate the position of my pendulum. I strongly recommend to build the cartesian axis system so that we always have the standard definition of the cylindrical coordinates. For example, the ϕ angle must be, according to our definition, the deviation from the x axis, that's why I didn't call this axis z , I called it x , to have, once again, my ϕ angle which appears here, and which describes, in a very convenient way, the oscillation of the pendulum. I have also drawn, the vectors e_ρ and e_ϕ , of the reference frame associated to the cylindrical coordinates.

Notes

Summary



2m 22s



Coordonnées cylindriques :

$$(\rho, \phi, z)$$

Contraintes géométriques :

$$z = 0$$

$$\rho = \underline{\ell}$$

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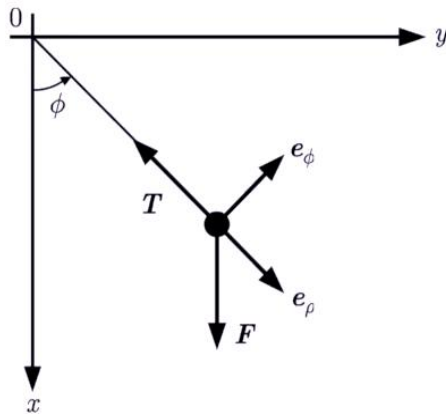
Now, in this problem, we have geometric constraints. Let's suppose that we stay in a vertical plane, if x is vertical, z is horizontal, we don't have any movement in the z direction, so we have a constraint on z , z is zero at all times. and ρ is the length of the wire which I have noted ℓ .

Notes

Summary



4m 13s



- Pesanteur
- Force de liaison

$$T = -T e_\rho$$

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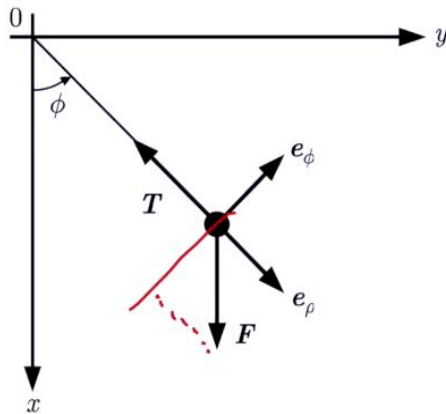
I now move on to the second step of our procedure, This is the establishment of the balance of forces. So, we have of course the force of gravity which I have called here F . I have the force, if we had a wire, obviously, we would have the force that the wire exerts, T . If we had instead drawn the arc of a circle here, we would have the same T , which would represent the reaction force of the circle on the material point. So I have, the gravity and the binding force, the only two forces which will intervene in this problem. The binding force, T , I will write it like this. By writing it like this, I am conforming to my drawing, I have drawn T in this direction, because my intuition tells me that T is directed in this direction. In this case it's obvious, if we extended the circle to the vertical upwards, we might be less sure of the sign of T . So, I come back to this position, T is directed in this direction. I'll assume T is positive and put a minus sign. Often students get confused and wonder which way they should put T . What I'm saying here is that you have to write the T here, that matches what you drew.

Notes

Summary



4m 41s



- Pesanteur
- Force de liaison

$$T = -T e_{\rho}$$

$$\mathbf{F} = F (\cos \phi e_{\rho} - \sin \phi e_{\phi})$$

In this way, if, at the end of the calculation, we find that T is positive, it means that we were right, this T was in the right direction, if at the end of the calculation, or in a certain regime and for certain positions, we find T negative, it means that the force T will be in the direction of e_{ρ} , in the other direction. So here, at this stage of solving the problem, we just put here a projection which corresponds to what we have drawn, there. For the force of gravity, at this point, I'm not assuming, I'm not saying F equals mg , I'm just going to write, F . I have to project this force onto the coordinate system associated with the cylindrical coordinates. Here I have angle ϕ , so I have $\cos \phi$ in the direction e_{ρ} . I must have $\sin \phi$, with a minus sign, because the projection of F will be on the opposite side to the direction given by e_{ϕ} . e_{ϕ} is always in the direction of increasing ϕ . So I put a minus sign here, my signs are correct by inspection of the graph, as long as my angles are acute.

Notes

Summary





$$\mathbf{a} = (\ddot{\rho} - \rho\dot{\phi}^2) \mathbf{e}_\rho + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}) \mathbf{e}_\phi + \ddot{z} \mathbf{e}_z$$

Contraintes géométriques (liaisons)

$$\rho = \ell \implies \dot{\rho} = \ddot{\rho} = 0$$

$$z = 0 \implies \dot{z} = \ddot{z} = 0$$

$$\mathbf{a} = (-\ell\dot{\phi}^2) \mathbf{e}_\rho + (\ell\ddot{\phi}) \mathbf{e}_\phi$$

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Next step in my process: the kinematics. Here, we want to use cylindrical coordinates. So, I'll look in the form for the expression of the vector acceleration projected onto the cylindrical coordinate frame. I have to apply the geometrical constraints, on the one hand, the coordinate ρ is a constant which is ℓ , the length of the wire. So here, $\dot{\rho}$ and $\ddot{\rho}$ are zero. This term is zero, this term is zero too. On the other hand, we are in the vertical plane, so z equals 0, \dot{z} is null, this term is null. I have two terms left to my vector acceleration, and you remember that we are on a circle, we have a term in \mathbf{e}_ρ , it's a centripetal acceleration, which is worth $\ell \dot{\phi}^2$, directed towards the center of the circle, and the $\ell \ddot{\phi}$ is along \mathbf{e}_ϕ , \mathbf{e}_ϕ is tangent to the circle. We have a tangential acceleration which is $\ell \ddot{\phi}$.

Notes

Summary



7m 49s



Dynamique : $\mathbf{F} + \mathbf{T} = m\mathbf{a}$

$$-m\ell\dot{\phi}^2 = F \cos \phi - T$$

$$m\ell\ddot{\phi} = -F \sin \phi$$

détermine le mouven

Remarque d'importance historique :
si le mouvement est indépendant de la masse, alors :

$$F = m g$$

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Now we can derive the equations of motion. We will use Newton's second law, $F = ma$. Here we have two forces, we have gravity, we have the binding force, and here is mass times acceleration. We have projected all the vector quantities here, beforehand, so, we are ready to write the equations of motion, which are here. We have two unknowns in these two equations, there is ϕ , the ϕ function of T , and there is the t function of T . In principle, the gravity is known, I make now just this remark with regard to the observation of Galileo, Galileo observed that the dynamics of the pendulum did not depend on the mass. This equation will give us the dynamics, the ϕ of T , if we want that this equation does not depend on the mass, the mass must fall from the equation. So we have to write F which is m times g , at that moment, the m 's cancel on both sides of the equal sign, there is no more mass in that equation, and we have what we wanted, an equation of motion which is independent of the mass. Once we have found this equation of motion, let's say we know how to integrate it, we can find ϕ of T . ϕ of T we put it here, and in ϕ dot. And the other equation will give us T .

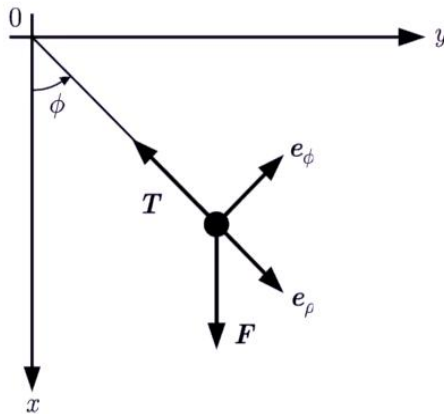
Notes

Summary



9m 15s

Petites oscillations autour d'un équilibre



$$\ddot{\phi} = -\frac{g}{\ell} \sin \phi$$

$$\ddot{x} = -\omega^2 x$$

Petits angles : $\ddot{\phi} = -\frac{g}{\ell} \phi$

Oscillateur harmonique !

Pulsation : $\omega = \sqrt{\frac{g}{\ell}}$

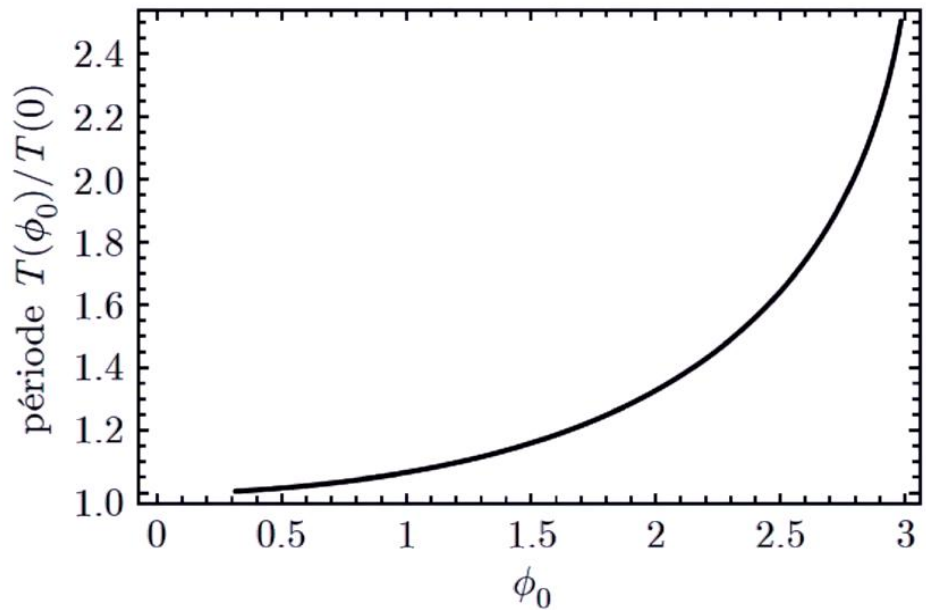
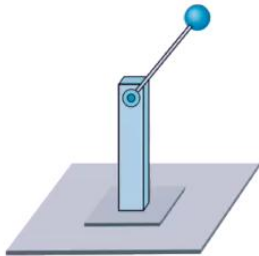
$$\omega = \frac{2\pi}{T} \quad T = 2\pi \sqrt{\frac{\ell}{g}}$$

Now I'm looking at the little oscillations, predicted by this equation of motion. So, I guess, the pendulum is down here, and so phi is small. Phi small, we'll model phi small by replacing the sine phi with phi. For small angles, my equation, in the limit of small angles, my equation of motion becomes like this. Meanwhile, I pause to give you time to recognize this differential equation. So this differential equation, it's the differential equation of a harmonic oscillator. We introduced the harmonic oscillator for a mass attached to a spring, so we had a differential equation of this form for a cartesian coordinate, here, we have an angular coordinate, but from the mathematical point of view, we have the same form of equation, so we'll say we have an equivalent motion. We have a harmonic oscillator, whose pulsation is omega, given by root of g on l. In the harmonic oscillator, we had something of the type $x \ddot{=} -\omega^2 x$. And we had called omega the pulsation. So here we have omega, which is the root of g over l. I recall that the omega pulsation, is 2 pi over the period, so we get, for the period, 2 pi times square root of l over g. Note that g is in meters per second squared, here we have meters, the square root gives us seconds, that is indeed the unit of a period.

Notes

Summary





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Now, I just considered the limit of small angles, what happens at large angles? So, I can imagine a pendulum that is constructed in the following way: you have a material point, linked to the fixed point by a rigid bar, we assume the bar is light, and we can ignore its mass. When you do the experiment, as you can see on the videos, you can see very clearly that if you start from very high, the period you observe is much bigger than the period you have when you have the small oscillations around the equilibrium position. I can make a drawing using the theoretical model, I have drawn here the amplitude ϕ_0 which is the amplitude of the oscillation, so, this is the initial amplitude. Here, I plot the period, normalized, by the period at the limit of the small oscillations. So here, we have a small oscillation, the function tends to one, and what we see is that if we have large oscillations, three is close to π , so we're up here, we have, practically, a period that is twice as large. So it's very clearly different, the period of the mathematical pendulum depends on the amplitude we give to this pendulum.

Notes

Summary



12m 53s

Intégration de l'équation du mouvement

$$\ddot{\phi} = -\frac{g}{\ell} \sin \phi$$

$$\ddot{\phi} \dot{\phi} = -\frac{g}{\ell} \sin \phi \dot{\phi}$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\phi}^2 \right) = \frac{d}{dt} \left(\frac{g}{\ell} \cos \phi \right)$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi \right) = 0$$

If now we want to obtain this curve, so the behaviour of the pendulum at any angle, we have to start again from the equation of the initial motion, that we had found in the general case, and we have to try to integrate it. We find here a differential equation which has the following form: here we have a function of phi, and there we have the second derivative of phi. In such a case, we can do the following manipulation: I multiply the equation by phi point, and now, when I look at this term, I recognize the derivative with respect to time of phi point squared, to within one coefficient. Similarly, here I have less sin phi, phi point, I recognize the derivative with respect to time of cos phi. This is what I wrote here, with the coefficients correctly placed. Now, I caution you, it often happens that a student who sees that these two derivatives are equal, feels like saying that this term is equal to that one. This is not correct, you have to look at it like this: you have d over dt, of the derivative of a half of phi square point, minus g over l, cos phi. That sucks. So I'm saying that's what's a constant. So don't get me wrong, if two terms have equal derivatives, that means they are equal, give or take a constant.

Notes

Summary



Intégration de l'équation du mouvement

$$\ddot{\phi} = -\frac{g}{\ell} \sin \phi$$

$$\ddot{\phi} \dot{\phi} = -\frac{g}{\ell} \sin \phi \dot{\phi}$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\phi}^2 \right) = \frac{d}{dt} \left(\frac{g}{\ell} \cos \phi \right)$$

$$\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi = \text{constante} = \underbrace{\frac{1}{2} \dot{\phi}_0^2}_0 - \frac{g}{\ell} \cos \phi_0$$

I write it like this. This constant, it must be the value of one half of phi point squared, minus g on l, cos phi, at any other time. In particular at time t equals 0, then I'll note the value of phi point at time t equals 0, phi point index 0. And there I'll put cosine of phi 0, the value of phi, at t equals 0. Now I suppose that, I drop my pendulum by an angle phi 0, but with a zero initial velocity, so I suppose that this term is zero, I'm left with this term, that's what I wrote here.

Notes

Summary



Intégration de l'équation du mouvement

$$\ddot{\phi} = -\frac{g}{\ell} \sin \phi$$

$$\ddot{\phi} \dot{\phi} = -\frac{g}{\ell} \sin \phi \dot{\phi}$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\phi}^2 \right) = \frac{d}{dt} \left(\frac{g}{\ell} \cos \phi \right)$$

$$\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi = \text{constante}$$

$$\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi = -\frac{g}{\ell} \cos \phi_0$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{\frac{2g}{\ell} (\cos \phi - \cos \phi_0)}$$

$$dt = \frac{d\phi}{\sqrt{\frac{2g}{\ell} (\cos \phi - \cos \phi_0)}}$$

And now, I can finish the integration with some algebraic manipulations. First, I do the following thing: I write phi square point equals 2 g on l, times cos phi, minus cos phi 0. Then I take the square root, so I'll do it, like this. Phi point, it's worth d phi over dt. So I can write dt equal, I pass the dt on that side of the equal sign, d phi, over the root of 2g over l, times the root of cos phi, minus cos phi 0. That's what I wrote on the line below.

Notes

Summary



Intégration de l'équation du mouvement

$$\ddot{\phi} = -\frac{g}{\ell} \sin \phi$$

$$\ddot{\phi} \dot{\phi} = -\frac{g}{\ell} \sin \phi \dot{\phi}$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\phi}^2 \right) = \frac{d}{dt} \left(\frac{g}{\ell} \cos \phi \right)$$

$$\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi = \text{constante}$$

$$\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi = -\frac{g}{\ell} \cos \phi_0$$

$$dt = \sqrt{\frac{\ell}{2g}} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}$$

Equation horaire inversée

$$t = \int_{\phi_0}^{\phi} \frac{d\phi'}{\sqrt{\cos \phi' - \cos \phi_0}} \left(\sqrt{\frac{\ell}{2g}} \right)$$

Intégrale elliptique

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I have the root of $2g$ on ℓ coming here, on the left, I have only an expression that depends on time and here, only an expression that depends on ϕ . I can now integrate, in the following way, I will integrate between time t equals 0, and some time t , and correspondingly, I integrate between the value of ϕ , the integration is over ϕ , I integrate, from the value of ϕ when t was equal to 0, this was ϕ_0 , and then some ϕ , function of time. The integral of the left member is trivial, so I have t equal to that integral. So, I integrated, but there is a little difficulty, it's that, normally, what we would call the time equation, it would be ϕ of t , and we see that here, what we get, is t of ϕ . I don't go any further because there is a difficulty, this integral doesn't have a simple analytical solution, it's what we call an elliptic integral. So we see that we start from an extremely simple problem, the mathematical pendulum, and we run into difficulties at the time of the integration. Any spreadsheet, or any integration program can calculate this, but I can't give an analytical expression. I can however use this formula to calculate the period.

Notes

Summary



Intégration de l'équation du mouvement

$$\ddot{\phi} = -\frac{g}{\ell} \sin \phi$$

$$\ddot{\phi} \dot{\phi} = -\frac{g}{\ell} \sin \phi \dot{\phi}$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\phi}^2 \right) = \frac{d}{dt} \left(\frac{g}{\ell} \cos \phi \right)$$

$$\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi = \text{constante}$$

$$\frac{1}{2} \dot{\phi}^2 - \frac{g}{\ell} \cos \phi = -\frac{g}{\ell} \cos \phi_0$$

$$dt = \sqrt{\frac{\ell}{2g}} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}$$

Equation horaire inversée

$\phi(t)$

$$\frac{T}{2} = \int_{\phi_0}^{-\phi_0} \frac{d\phi'}{\sqrt{\cos \phi' - \cos \phi_0}} \left(\sqrt{\frac{\ell}{2g}} \right)$$

Intégrale elliptique

Mécanique | 2013 51

If I go from ϕ_0 to $-\phi_0$, I have made a half beat, so the time, here, will become the half period. We see here that we have units of time, there, we find a coefficient root of ℓ on g which was proportional to the period that we had T at ϕ_0 , it's not t equals 0, there are factors 2π and there is this root of 2 that appeared, but you see why in the previous drawing, it was easy and natural to express the period T for an amplitude ϕ_0 , in terms of the period for an amplitude 0, or close to 0. This method of integration can be used in a large number of cases because when we do mechanics, we often have an equation of motion that has this structure. So, it can be very, very useful. That's it, I'm done with the example of the mathematical pendulum. This course, I declared it to be a know-how course, to learn to solve problems, you have to roll up your sleeves and do problems by yourself. I stop my lesson here, I invite you to go and see the series of exercises, where you will find problems with material points forced to move on surfaces, or particular lines.

Notes

Summary

