

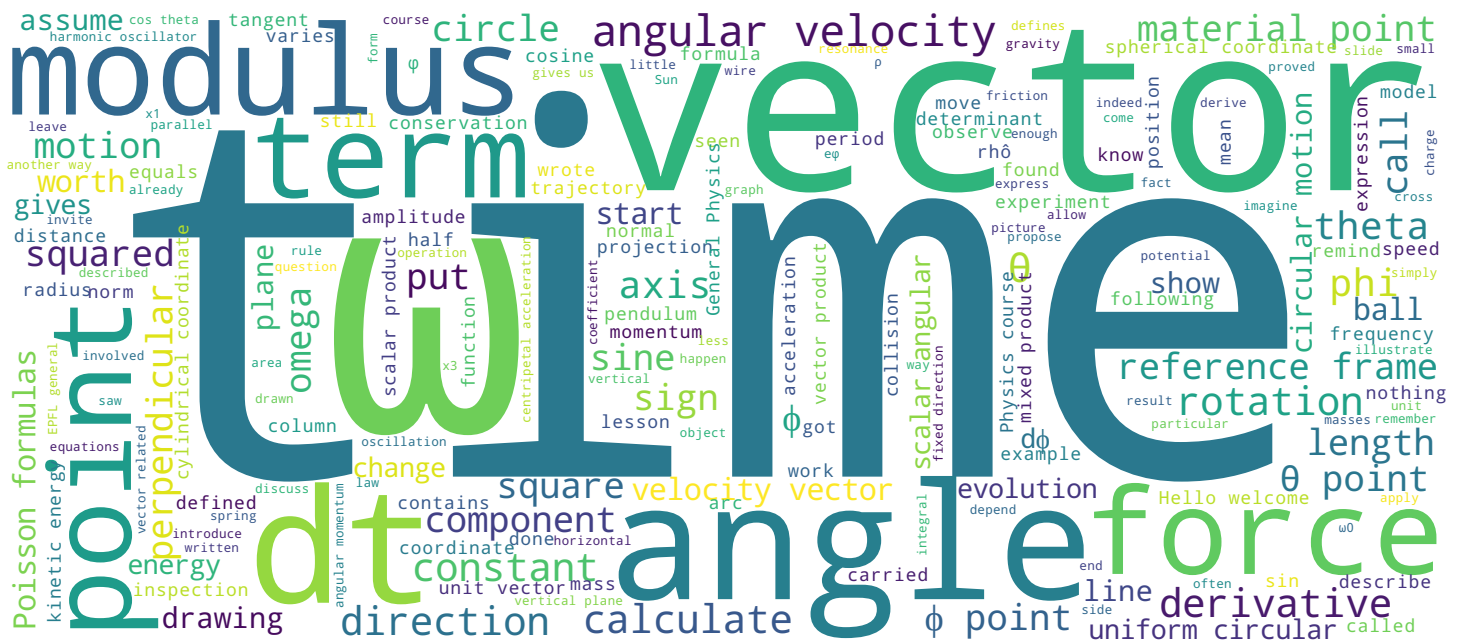


## Vitesse angulaire

## Mécanique, cours 8.2

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<http://go.epfl.ch/traite-meca-2-8>



EPFL

## Video





- Formules de Poisson : rotation
- Le vecteur vitesse angulaire
  - sa direction
  - son module
- Applications

Mécanique | 2013 7

Hello, welcome to the ÉPFL General Physics course. In this lesson, we discuss the question of describing rotations, in particular the kinematics when rotations for a material point are involved. So, we saw that we could express the time derivatives of the unit vectors of a reference frame using a single expression, which we called the Poisson formulas, which involved a vector  $\omega$ , called angular velocity. Now we need to understand the geometric interpretation of this angular velocity vector. So, first, we'll see that Poisson's formulas describe a rotation. This will allow us to realize that the angular velocity vector we have defined is in the direction of the axis of rotation, and that its modulus is nothing else than the scalar angular velocity, that we had already defined. We will see some applications of the formalism, to illustrate how to use this new tool.

Notes

Summary



0m 04s

# Les formules de Poisson décrivent une rotation

$$\frac{d\hat{e}_i}{dt} = \boldsymbol{\omega} \wedge \hat{e}_i \quad (i = 1, 2, 3) \implies \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge \mathbf{r}$$

$\boldsymbol{\omega}$  : vitesse angulaire

Pour tout  $\mathbf{r} \parallel \boldsymbol{\omega}$   $\frac{d\mathbf{r}}{dt} = 0$   $\boldsymbol{\omega}$  définit une direction fixe

Les angles et les longueurs sont conservées

$$\frac{d}{dt} (\mathbf{r}_1 \cdot \mathbf{r}_2) = 0$$

I start with the Poisson formulas, which I can generalize to any vector related to the reference frame  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ , it gives us that formula there, the Poisson formulas, the case of a vector related to  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ . With  $\boldsymbol{\omega}$  the angular velocity. First observation, if  $\mathbf{r}$  is parallel to  $\boldsymbol{\omega}$ ,  $d\mathbf{r}$  over  $dt$  is zero. So this equation there describes an evolution of the vectors  $\mathbf{r}$  belonging to the reference frame which leaves a whole series of vectors unchanged, these are the vectors along  $\boldsymbol{\omega}$ . In other words, for any  $\mathbf{r}$  parallel to  $\boldsymbol{\omega}$ ,  $\mathbf{r}$  is constant, so  $\boldsymbol{\omega}$  defines a fixed direction, unchanged by the operation that is described by this evolution equation. So, first thing, we have a fixed direction. Now, I'm going to show that, under this operation, the angles and the lengths are preserved. I consider 2 vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$  related to the reference frame, and I'm going to prove to you that the derivative with respect to time of this scalar product is zero.

Notes

Summary



# Les formules de Poisson décrivent une rotation

$$\frac{d\hat{e}_i}{dt} = \omega \wedge \hat{e}_i \quad (i = 1, 2, 3) \implies \frac{d\mathbf{r}}{dt} = \omega \wedge \mathbf{r}$$

$\omega$  : vitesse angulaire

Pour tout  $\mathbf{r} \parallel \omega$   $\frac{d\mathbf{r}}{dt} = 0$   $\omega$  définit une direction fixe

Les angles et les longueurs sont conservées

$$\frac{d}{dt} (\mathbf{r}_1 \cdot \mathbf{r}_2) = 0 = \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \frac{d\mathbf{r}_2}{dt} = (\omega \wedge \mathbf{r}_1) \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot (\omega \wedge \mathbf{r}_2)$$

When I have done this, I will have proved that the angles and lengths are conserved. Why? Because if I take  $\mathbf{r}_2 = \mathbf{r}_1$ , I will have the modulus of  $\mathbf{r}_1$  squared, and so I will have found that the modulus squared is a constant, so the modulus is constant. If now I take  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , 2 unit vectors, the scalar product is worth the cosine of the angle between the 2 vectors, and here I have that the cosine is constant, so the angle between the 2 vectors is constant. That's what I announced here. So let's look at this derivative, we're going to carry the derivative over time, first on  $\mathbf{r}_1$ , then on  $\mathbf{r}_2$ , that gives us these 2 terms. There's  $d$  from  $\mathbf{r}_1$  over  $dt$ ,  $d$  from  $\mathbf{r}_2$  over  $dt$ . For  $d$  of  $\mathbf{r}_1$  over  $dt$ , I'll use this rule, like this, for  $d$  of  $\mathbf{r}_2$  over  $dt$  too, and there you go. Here I observe a mixed product: this is a vector product, it gives a vector, and here I have a scalar product, so I have a number.

Notes

Summary



# Les formules de Poisson décrivent une rotation

$$\frac{d\hat{e}_i}{dt} = \boldsymbol{\omega} \wedge \hat{e}_i \quad (i = 1, 2, 3) \implies \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge \mathbf{r}$$

$\boldsymbol{\omega}$  : vitesse angulaire

Pour tout  $\mathbf{r} \parallel \boldsymbol{\omega}$   $\frac{d\mathbf{r}}{dt} = 0$   $\boldsymbol{\omega}$  définit une direction fixe

Les angles et les longueurs sont conservées

$$\begin{aligned} \frac{d}{dt} (\mathbf{r}_1 \cdot \mathbf{r}_2) &= 0 = \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \frac{d\mathbf{r}_2}{dt} = (\boldsymbol{\omega} \wedge \mathbf{r}_1) \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_2) \\ &= -(\mathbf{r}_1 \wedge \boldsymbol{\omega}) \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_2) = 0 \end{aligned}$$

L'évolution est caractéristique d'une rotation !

Mécanique | 2013 19

And this mixed product, we had seen, can be written or can be calculated as the determinant, in which I put the components of  $\mathbf{r}_1$  on the first column, the components of  $\boldsymbol{\omega}$  on the second column, the components of  $\mathbf{r}_2$  on the third column. Here I have  $\boldsymbol{\omega} \wedge \mathbf{r}_1$ , the cross product is defined by a determinant, if I cross 2 columns, I change the sign of the determinant, so when I cross  $\boldsymbol{\omega}$  and  $\mathbf{r}_1$ , like this, I change the sign. And I have, again here, in the first term, a mixed product which I can calculate as the determinant, in which I put  $\mathbf{r}_1$  in the first column,  $\boldsymbol{\omega}$  in the second,  $\mathbf{r}_2$  in the third, exactly as here. And I have a minus sign, I have the same determiner twice, once with a + sign, once with a - sign, so it's null. Thus, I have proved that this evolution there is characteristic of a transformation that leaves one direction fixed and all angles and lengths conserved, it is a rotation.

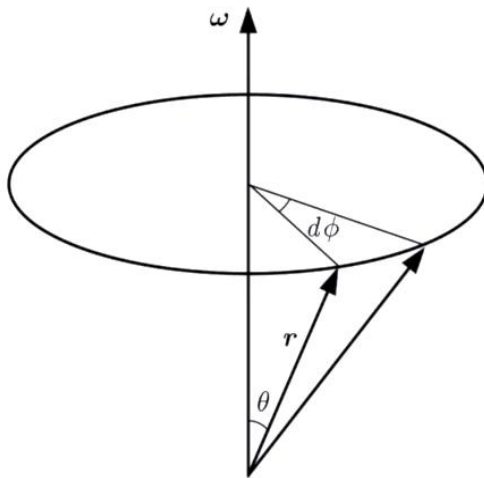
Notes

Summary



3m 57s

# Module du vecteur de vitesse angulaire



$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge \mathbf{r}$$

$$\underbrace{|\mathbf{r}(t + dt) - \mathbf{r}(t)|}_{d\mathbf{r}} = |\mathbf{r}| |\boldsymbol{\omega}| dt \sin \theta$$

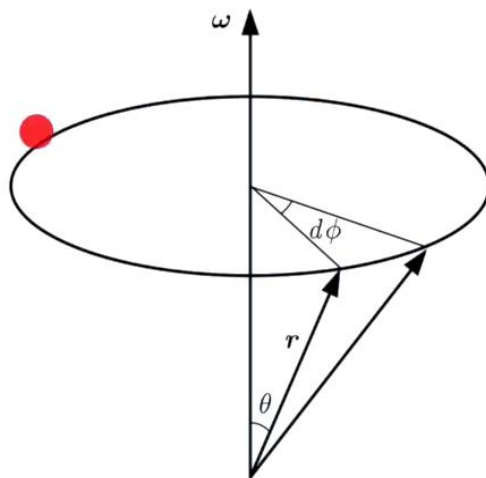
Now we still need to determine the modulus of the angular velocity vector, the modulus of  $\boldsymbol{\omega}$ . So, I'll make a geometric representation of what this relationship predicts. Here you have the vector  $\boldsymbol{\omega}$ , which is carried by the axis of rotation, I assume a vector  $\mathbf{r}$ , like this. And this relation there tells us:  $d\mathbf{r}$  is  $\boldsymbol{\omega} \wedge \mathbf{r}$  times  $dt$ . So if I calculate the  $\mathbf{r}$  at a time  $t + dt$ , I'll have a picture like this, here's  $\mathbf{r}$  by  $t + dt$ , and here's the  $d\mathbf{r}$ . Now I have  $d\mathbf{r}$ , predicted by that equation there, which is worth, sorry, the modulus of  $d\mathbf{r}$  is worth the modulus of  $\boldsymbol{\omega} \wedge \mathbf{r}$  times  $dt$ . Here's the  $dt$ , The modulus of  $\boldsymbol{\omega} \wedge \mathbf{r}$ , we've seen, is the modulus of  $\boldsymbol{\omega}$  times the modulus of  $\mathbf{r}$  times the sine of the angle between the 2. So in the drawing I decided to call  $\theta$  the angle between  $\mathbf{r}$  and  $\boldsymbol{\omega}$ . So I have a sine of  $\theta$  appearing here.

Notes

Summary



# Module du vecteur de vitesse angulaire



$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge \mathbf{r}$$

$$|\mathbf{r}(t+dt) - \mathbf{r}(t)| = |\mathbf{r}| |\boldsymbol{\omega}| dt \sin \theta$$

$$|\mathbf{r}(t+dt) - \mathbf{r}(t)| = |\mathbf{r}| |d\phi| \sin \theta$$

$$|\boldsymbol{\omega}| = \left| \frac{d\phi}{dt} \right| = |\dot{\phi}|$$

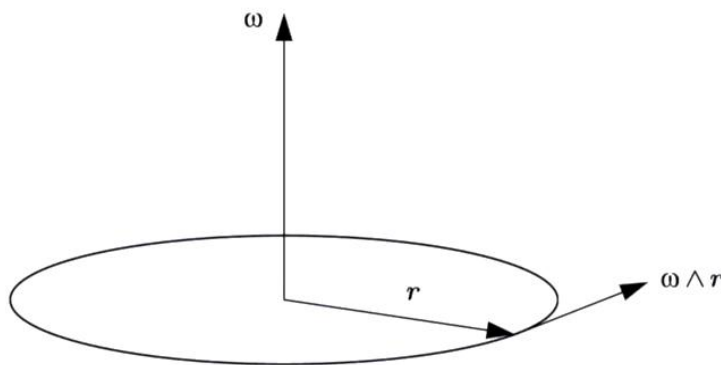
Now, I can look at this drawing another way, I can consider that the end of the vector  $\mathbf{r}$  has rotated on this circle by an angle  $d\phi$ . So, trigonometry tells me that the length of this arc, which is about  $d\mathbf{r}$  also; I have my  $d\mathbf{r}$  again, here, and I look up the modulus of  $d\mathbf{r}$ , that is about the length of this arc. We need to know this radius, this radius is modulus of  $\mathbf{r}$  times the sine of  $\theta$ . And the length of the arc is  $r \sin \theta$  times  $d\phi$ ,  $r \sin \theta$  times  $d\phi$ , that's what I wrote here. I compare these 2 expressions, I see that  $\boldsymbol{\omega} dt$  is  $d\phi$ . So the modulus of  $\boldsymbol{\omega}$  is worth the modulus of  $d\phi$  over  $dt$ , that is, the modulus of  $\dot{\phi}$  point, and that  $\dot{\phi}$  point there is what we had called the scalar angular velocity. So I just got that the modulus of that  $\boldsymbol{\omega}$  is nothing but the scalar angular velocity, that we had described for that motion there.

Notes

Summary



# Application : mouvement circulaire uniforme



$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge \mathbf{r}$$

$$\mathbf{a} = \frac{d}{dt} (\boldsymbol{\omega} \wedge \mathbf{r}) = \boldsymbol{\omega} \wedge \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})$$

Let's just look at the uniform circular motion. Here is the trajectory of a material point, which describes a uniform circular motion, this time I make a 3D drawing. To this motion corresponds a rotation whose axis is here. So I have a  $\boldsymbol{\omega}$  carried by the axis, I have drawn the  $\boldsymbol{\omega}$  like this. Now, my formula tells me that the  $d\mathbf{r}$  velocity on  $dt$  is  $\boldsymbol{\omega} \wedge \mathbf{r}$  for a rotation; and so I have the  $\boldsymbol{\omega} \wedge \mathbf{r}$ , by inspection of the drawing, I see that the  $\boldsymbol{\omega} \wedge \mathbf{r}$ , right hand rule, will be in this sense, like this. And this formula, given the vector product property, we know that  $\mathbf{v}$  is perpendicular to  $\mathbf{r}$ , since we are on a circle, so we have  $\mathbf{v}$  is tangent to the circle, since the  $\boldsymbol{\omega} \wedge \mathbf{r}$  is perpendicular to  $\mathbf{r}$ . I can now calculate the acceleration, and I assume I have uniform circular motion, so  $\boldsymbol{\omega}$  is constant. So when I calculate the acceleration, I calculate the derivative with respect to time of the velocity,  $\boldsymbol{\omega} \wedge \mathbf{r}$ . So I only have one term  $\boldsymbol{\omega} \wedge (d\mathbf{r} \text{ over } dt)$ .

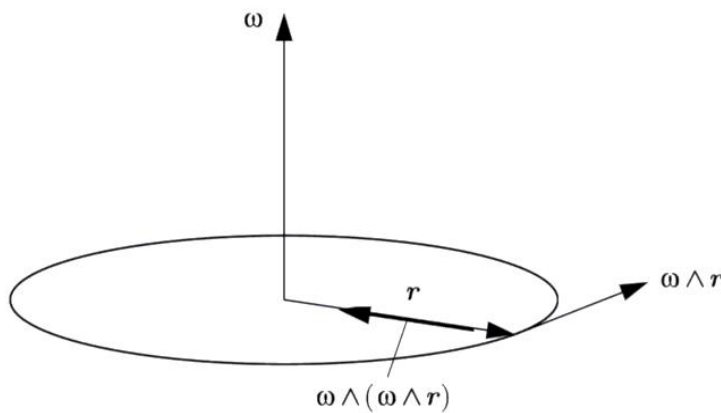
Notes

Summary





# Application : mouvement circulaire uniforme



$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge \mathbf{r}$$

$$\mathbf{a} = \frac{d}{dt} (\boldsymbol{\omega} \wedge \mathbf{r}) = \boldsymbol{\omega} \wedge \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})$$

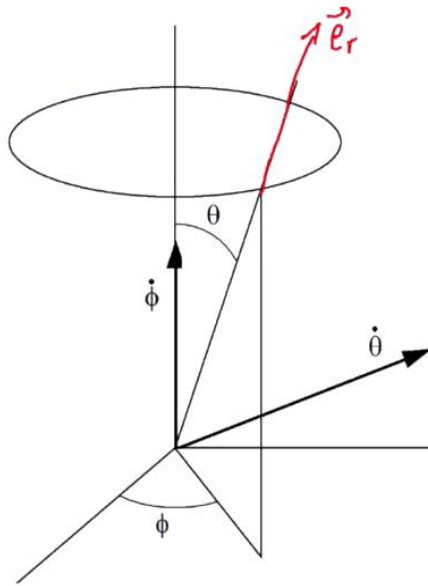
$|\boldsymbol{\omega}| = \omega$  vitesse angulaire scalaire

For the  $d\mathbf{r}$  over  $dt$ , I apply the formula again, I have  $\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})$ . You have to be careful to put the parentheses, otherwise this expression would not be defined. By inspection of the drawing,  $\boldsymbol{\omega}$ , vector product with  $\boldsymbol{\omega} \wedge \mathbf{r}$  gives a vector in that direction, opposite to  $\mathbf{r}$ , that's centripetal acceleration. So here I have a very elegant expression of the centripetal acceleration for uniform circular motion.  $\mathbf{a} = \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})$  The modulus of  $\boldsymbol{\omega}$ , again, is the scalar angular velocity that we had introduced when we first looked at uniform circular motion.

Notes

Summary





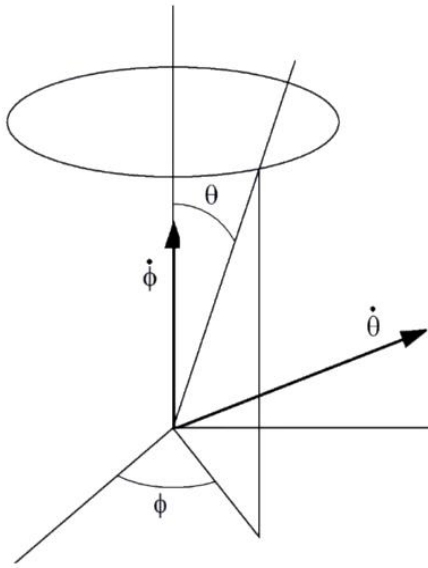
$$\frac{d\hat{e}_r}{dt} = \dot{\phi} \wedge \hat{e}_r + \dot{\theta} \wedge \hat{e}_r$$

I take another example, here are the 2 angles that define the spherical coordinates. There is still this length there, but as for the angles, we have the 2 angles here. The  $\phi$  angle describes a rotation about this axis, so I have an angular velocity associated with the  $\phi$  angle, which is  $\dot{\phi}$  **point**, a vector I'm going to write down  $\dot{\phi}$  **point**, carried by the axis. The angle  $\theta$  defines a rotation in the vertical plane that contains these lines, hence the axis of rotation is normal to that plane, roughly in this direction, like this. This is the angular velocity vector associated with the rotation of angle  $\theta$ . Let's examine, for example the evolution over time of the  $\mathbf{e}_r$  vector. I remind you, the vector  $\mathbf{e}_r$  is like this. Now, I can use Poisson formulas, to say that the derivative with respect to time of the vector  $\mathbf{e}_r$  must be the  $\boldsymbol{\omega} \wedge \mathbf{e}_r$ , with the  $\boldsymbol{\omega}$  for that particular rotation, so  $\dot{\phi}$  **point**, and then the  $\boldsymbol{\omega} \wedge \mathbf{e}_r$  for the rotation defined by the  $\theta$  angle, so it's from  $\dot{\theta}$  **point**. By inspection of the graph, I can get these 2 terms.

Notes

Summary





$$\begin{aligned}\frac{d\hat{e}_r}{dt} &= \dot{\phi} \wedge \hat{e}_r + \dot{\theta} \wedge \hat{e}_r \\ &= \dot{\phi} \sin \theta \hat{e}_\phi + \dot{\theta} \hat{e}_\theta\end{aligned}$$

I'm going to have a vector here that is perpendicular to  $\mathbf{e}_r$ , and perpendicular to  $\phi$  point, so it's perpendicular to that plane. So it's in the direction of  $\mathbf{e}_\phi$ , you remember that  $\mathbf{e}_\phi$  is normal to the vertical plane that contains this line and that one. So I will have a term like this. There's my  $\mathbf{e}_\phi$ . What's the norm of that vector there? Well it's the norm of  $\phi$  point times the norm of  $\mathbf{e}_r$  (that's 1) times the sine of the angle between the 2, so sine  $\theta$ . That's what I have here. For the other term, inspection of the graph, I lost my  $\mathbf{e}_r$  but I'll gladly draw it again, here  $\theta$  point, we have to do  $\theta$  point  $\wedge \mathbf{e}_r$ , this time we have a right angle,  $\mathbf{e}_r$  belongs to the plane that contains this line, this one and that one.  $\theta$  point is perpendicular. Again,  $\theta$  point is in the direction of  $\mathbf{e}_\phi$ . Therefore,  $\theta$  point  $\wedge \mathbf{e}_r$  must be in the direction of  $\mathbf{e}_\theta$ , I remind you that the  $\mathbf{e}_\theta$  is pointing in that direction there, what I wrote here.  $\theta$  point times  $\mathbf{e}_\theta$  This is another way to get the  $d$  from  $\mathbf{e}_r$  on  $dt$ , we had done it with an argument that did not use Poisson formulas yet.

Notes

Summary





- Le vecteur de vitesse angulaire est sur l'axe de rotation
- Son module est la vitesse angulaire scalaire
- Si on a plusieurs rotations, on somme les vitesses angulaires

Mécanique | 2013 43

I summarize: we got that the angular velocity vector is on the axis of rotation, that its modulus is nothing but the scalar angular velocity we had already met, and if we have several rotations, you will have noticed, it is enough, very naturally, to sum the angular velocity vectors.

Notes

Summary



14m 07s