



- Evolution d'un repère
- Matrice
- Vitesse angulaire vectorielle
- Formules de Poisson

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Hello, welcome to the EPFL general physics course. In this lesson, I would like to tackle the question of describing rotations mathematically. Fortunately, for what concerns the dynamics or kinematics of point masses, it will be enough to introduce an angular velocity vector to describe what we need. The question of rotations comes up already because we have defined reference frames associated with cylindrical and spherical coordinates that evolve in time, and we need to see how to calculate the time derivatives of the reference frame unit vectors. We will see that, we have to introduce a matrix, but that this matrix has particular properties which will allow us to simplify the expression of the problem, and use simply an angular velocity that we will define here. This will lead us to what I call the Poisson formulas, formulas that I will use often in the rest of the course.

Notes

Summary



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Evolution des vecteurs unités d'un repère

$(A, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ A fixe

$$\frac{d\hat{e}_1}{dt} = E_{21}\hat{e}_2 + E_{31}\hat{e}_3 \quad (\text{normal à } \hat{e}_1)$$

So let's look at the question of the evolution of the unit vectors of a reference frame. I imagine a reference frame defined by the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$. I propose to imagine a situation with a fixed A . In fact, these three vectors are free vectors, so the fact that A is fixed is not very important, but it allows us to imagine these three vectors in motion around the point A . And this motion is a rotation, as we will see. If I apply the rule that we have already given ourselves, those we observed when we were studying the uniform circular motion, we saw that, for any vector of constant norm, its derivative with respect to time is perpendicular to this vector. So the derivative of \hat{e}_1 with respect to time is perpendicular to \hat{e}_1 . What I have expressed here is that in a very general way, the derivative of \hat{e}_1 with respect to time has a component along \hat{e}_2 and \hat{e}_3 . One of the components can be zero, but, in general, I have both possibilities. For the coefficients, I gave them a particular notation. With the second index referring to the vector that I derive with respect to time.

Notes

Summary



Evolution des vecteurs unités d'un repère

$$(A, \hat{e}_1, \hat{e}_2, \hat{e}_3) \quad A \text{ fixe}$$

$$\frac{d\hat{e}_1}{dt} = E_{21}\hat{e}_2 + E_{31}\hat{e}_3 \quad (\text{normal à } \hat{e}_1)$$

$$\frac{d\hat{e}_i}{dt} = \sum_{j=1}^3 E_{ji}\hat{e}_j \quad (\text{avec } E_{ii} = 0)$$

$$\hat{e}_i \cdot \hat{e}_k = \delta_{ik} \implies \frac{d}{dt}(\hat{e}_i \cdot \hat{e}_k) = 0$$

$$0 = \sum_{j=1}^3 E_{ji}\hat{e}_j \cdot \hat{e}_k + \hat{e}_i \cdot \sum_{j=1}^3 E_{jk}\hat{e}_j = E_{ki} + E_{ik}$$

The reason I use this funny notation, is because I'm now going to construct a matrix, which comes from the following consideration: Let me generalize this law that I wrote for the vector \hat{e}_1 , I now write it, this evolution law, for the vector \hat{e}_i , i equals 1, 2, 3. Here I have coefficients E_{ji} which intervene, and I must simply take the precaution that E_{ii} is zero. This comes from the fact that the derivative of \hat{e}_i with respect to time is perpendicular to \hat{e}_i . Now I want to express the fact that these vectors are orthogonal to each other. I write: \hat{e}_i scalar product with \hat{e}_k is either 1, if k equals i , or 0 if k is different from i , this is what the Kronecker symbol means. Whatever the result of this scalar product is, it is independent of time. So, I can write that the derivative with respect to time of this scalar product is zero. Now, I will operate the derivative, once on the first term, once on the second term. This gives me this: when I carry the derivative over \hat{e}_i , I apply this formula, so I have E_{ji} times \hat{e}_j , scalar product with \hat{e}_k , when I carry the derivative over \hat{e}_k , I have \hat{e}_i scalar product, with then, the derivative with respect to time of \hat{e}_k , I get it with that formula, replacing i by k .

Notes

Summary



Evolution des vecteurs unités d'un repère

$$(A, \hat{e}_1, \hat{e}_2, \hat{e}_3) \quad A \text{ fixe}$$

$$\frac{d\hat{e}_1}{dt} = E_{21}\hat{e}_2 + E_{31}\hat{e}_3 \quad (\text{normal à } \hat{e}_1)$$

$$\frac{d\hat{e}_i}{dt} = \sum_{j=1}^3 E_{ji}\hat{e}_j \quad (\text{avec } E_{ii} = 0)$$

$$\frac{d\hat{e}_i}{dt} = \sum_{j=1}^3 E_{ij}^T \hat{e}_j$$

$$\hat{e}_i \cdot \hat{e}_k = \delta_{ik} \implies \frac{d}{dt} (\hat{e}_i \cdot \hat{e}_k) = 0$$

$$0 = \sum_{j=1}^3 E_{ji}\hat{e}_j \cdot \hat{e}_k + \hat{e}_i \cdot \sum_{j=1}^3 E_{jk}\hat{e}_j = E_{ki} + E_{ik}$$

$$E^T = \begin{pmatrix} 0 & E_{12} & E_{13} \\ -E_{12} & 0 & E_{23} \\ -E_{13} & -E_{23} & 0 \end{pmatrix} \quad E^T = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

Une convention !

So here I have E_{jk} . E_{jk} times \hat{e}_j . And now the scalar product $\hat{e}_j \cdot \hat{e}_k$ is zero, except when j is k . At that point, E_{ki} remains. This is what I wrote here. Similarly on this side, when \hat{e}_j scalar product with... \hat{e}_j is only worth 1 when i equals j . So that j is equal to i , and we have E_{ik} . It is the E_{ik} that is here. So I arrive at the conclusion: E_{ki} equals minus E_{ik} . My matrix is antisymmetric. If I write the matrix in this form, in array form, I have, for the element $E_{1,2}$ here, on this side I'll have $E_{2,1}$ which is minus $E_{1,2}$, that's what I indicated. $E_{1,3}$, minus $E_{1,3}$, $E_{2,3}$, minus $E_{2,3}$ and of course we have 0 on the diagonal because if E_{ii} is to be equal to minus E_{ii} , E_{ii} must be zero. Now I introduce a particular notation, there is not much reason apparently, at this point in the calculation, to use this notation, but you will see that it is very useful. I decide to call $E_{1,2}$, ω_3 minus $E_{1,3}$ ω_2 , and $E_{2,3}$ I call it ω_1 . It's completely arbitrary at this point, but this writing is going to give us the angular velocity vector that I would like to introduce in this lesson. So this is a writing convention. It is this writing convention that requires that we always use direct bearings.

Notes

Summary



Définition : vecteur vitesse angulaire

Pour tout \mathbf{r} fixé dans le repère :

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt} (r_1 \hat{\mathbf{e}}_1 + r_2 \hat{\mathbf{e}}_2 + r_3 \hat{\mathbf{e}}_3) = \sum_i r_i \frac{d\hat{\mathbf{e}}_i}{dt} \\ &= \sum_i r_i \sum_j E_{ji} \hat{\mathbf{e}}_j = \sum_j \left(\sum_i E_{ji} r_i \right) \hat{\mathbf{e}}_j \\ \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} &= \begin{pmatrix} -\omega_3 r_2 + \omega_2 r_3 \\ \omega_3 r_1 - \omega_1 r_3 \\ -\omega_2 r_1 + \omega_1 r_2 \end{pmatrix}\end{aligned}$$

So, if I consider now a vector \mathbf{r} fixed in the reference frame. Be careful: velocities are measured with respect to the reference frame. The coordinate frame, for example the cylindrical and spherical coordinate frame, moves in time. It evolves in time. Now I can agree to consider a vector \mathbf{r} which is fixed in this coordinate frame, it doesn't mean that the coordinate frame is a reference frame. So I'll write my \mathbf{r} -vector, like this, this is my \mathbf{r} -vector, and so I'm assuming that the components of the \mathbf{r} -vector are not time dependent. This is how we're going to express the fact that \mathbf{r} is fixed in the reference frame. So, if the vector \mathbf{r} evolves in time, it's because the coordinate frame evolves in time. So when I compute $d\mathbf{r}/dt$, the derivative is on the unit vectors only. That's what I wrote here. Now I use the formula I gave for $d\hat{\mathbf{e}}_i/dt$, E_{ji} times $\hat{\mathbf{e}}_j$, sum over j , I group the terms into i . There's one here, there's another one here, I put the sum over i , all together, and so I have here the element, the component j of my vector, $d\mathbf{r}/dt$ is a vector, the component j of this vector it's this, it's the result of computing a matrix product, the matrix E_{ji} which is therefore the transpose of the matrix E that I have just defined, that's why I have here all the signs that are inverted, times the vector \mathbf{r} , whose components are r_1, r_2, r_3 .

Notes

Summary



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Pour tout \mathbf{r} fixé dans le repère :

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt} (r_1 \hat{\mathbf{e}}_1 + r_2 \hat{\mathbf{e}}_2 + r_3 \hat{\mathbf{e}}_3) = \sum_i r_i \frac{d\hat{\mathbf{e}}_i}{dt} \\ &= \sum_i r_i \sum_j E_{ji} \hat{\mathbf{e}}_j = \sum_j \left(\sum_i E_{ji} r_i \right) \hat{\mathbf{e}}_j \\ \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} &= \begin{pmatrix} -\omega_3 r_2 + \omega_2 r_3 \\ \omega_3 r_1 - \omega_1 r_3 \\ -\omega_2 r_1 + \omega_1 r_2 \end{pmatrix} = \boldsymbol{\omega} \wedge \mathbf{r} \\ \boldsymbol{\omega} &= \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad \mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \wedge \mathbf{r}\end{aligned}$$

This writing is equivalent to this one. Now, if I explain the terms, the components of the vector $d\mathbf{r}/dt$, I have minus omega 3 times r2, plus omega 2 times r3, that's this term, omega 3 r1 minus omega 1, r3, here, minus omega 2, r1, plus omega 1, r2, there. So, when you get to this point, you can see why I made the notation choice I did, because if now I define an omega vector with omega 1, omega 2, omega 3 as components, then what I just got here is nothing but omega cross r. In fact, if I look at the first component of this vector, so if I wanted to calculate the product with a determinant e1, e2, e3, you imagine here in your head, the determinant formed from the three columns, this column, that column and that column, the first component of the vector is omega 2 r3 minus omega 3 r2. That's it. The second one gives you omega 3 r1 minus omega 1 r3. There it is. The third one, omega 1 r2 minus omega 2 r1, there it is. So, I found $d\mathbf{r}/dt$ equals omega cross r for any vector r fixed in the reference frame. I can now particularize this formula by taking r1 equals 1, r2 zero, r3 zero. So I have the evolution of e1 in time. $d\mathbf{e}_1/dt$ equals omega cross e1. Same thing for the vectors e2 and e3.

Notes

Summary





$$\frac{d\hat{e}_i}{dt} = \boldsymbol{\omega} \wedge \hat{e}_i \quad (i = 1, 2, 3)$$

$\boldsymbol{\omega}$: vitesse angulaire

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This is what I call the Poisson formulas. For this frame, the evolution of the vectors in the frame can be described by this omega here, and there is the same omega for the three vectors, i equals 1, 2, 3. I have d of \hat{e}_i over dt, which is omega, vector product with \hat{e}_i , omega I will call the angular velocity.

Notes

Summary



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