

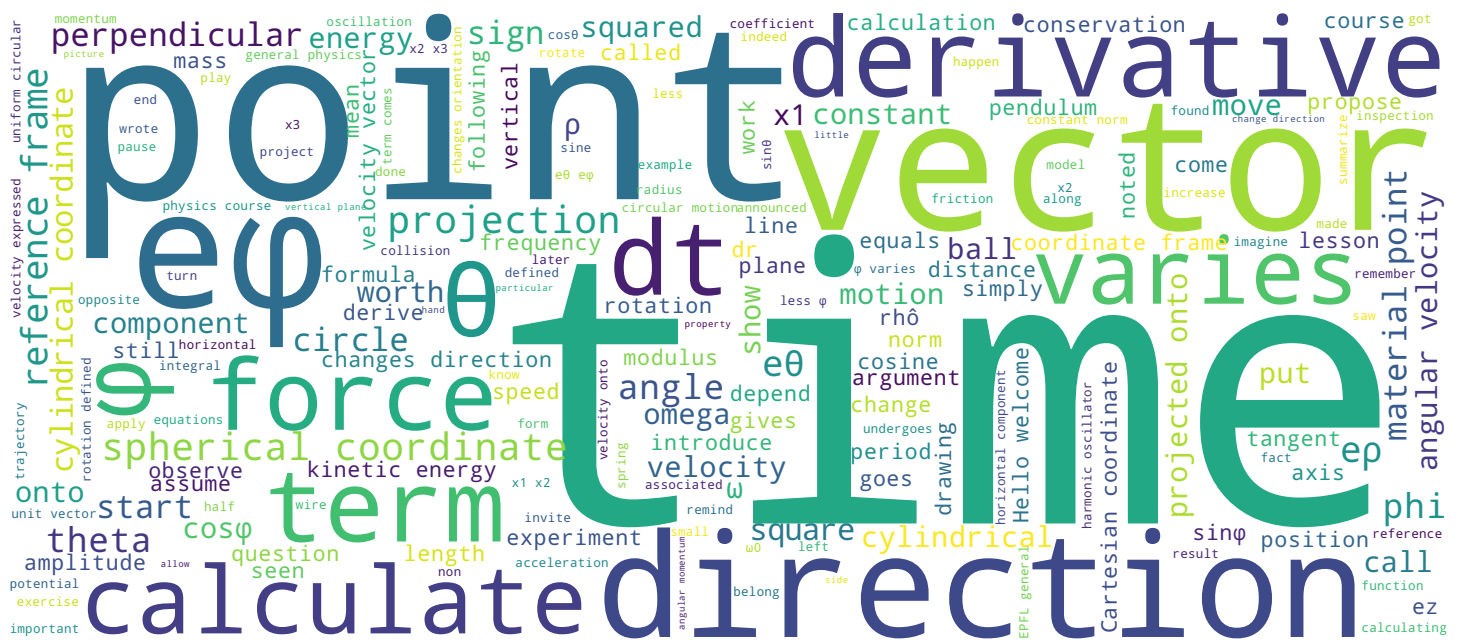


<http://go.epfl.ch/traite-meca-2-7>

Vitesse en coord. cylindriques et sphériques

Mécanique, cours 7.2

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Video



Vitesse en coord. cylindriques et sphériques



Vitesse vectorielle, projection sur le repère des coordonnées :

- cylindriques
- sphériques

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Hello! Welcome to the EPFL general physics course. In this lesson, I introduce the cylindrical and spherical coordinates. And I want to see how we will do the kinematics of the material point expressed in cylindrical and spherical coordinates. So...the velocity is a vector quantity. And what I want is the projections of the velocity, expressed in cylindrical or spherical coordinates. and projected on the cylindrical or spherical coordinate system.

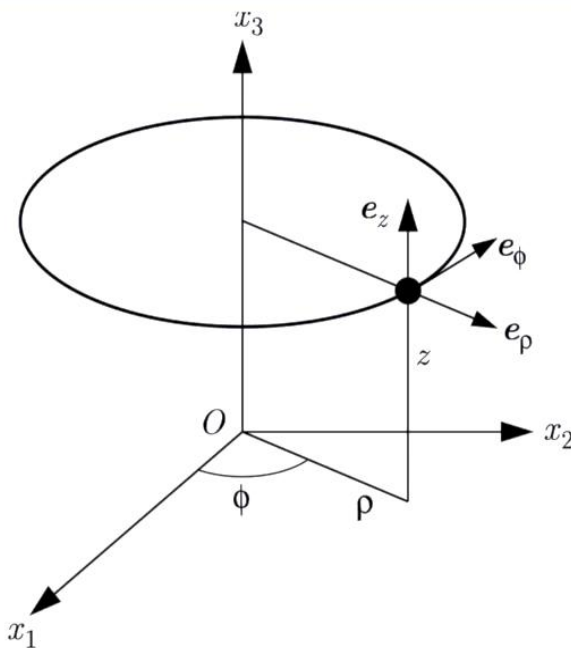
Notes

Summary



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Vitesse projetée sur le repère des c. cylindriques



$$x_1 = \rho \cos \phi$$

$$x_2 = \rho \sin \phi$$

$$x_3 = z$$

$$\mathbf{r} = \rho \hat{\mathbf{e}}_\rho + z \hat{\mathbf{e}}_z$$

$$\mathbf{v} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\phi} \mathbf{e}_\phi + \dot{z} \mathbf{e}_z$$

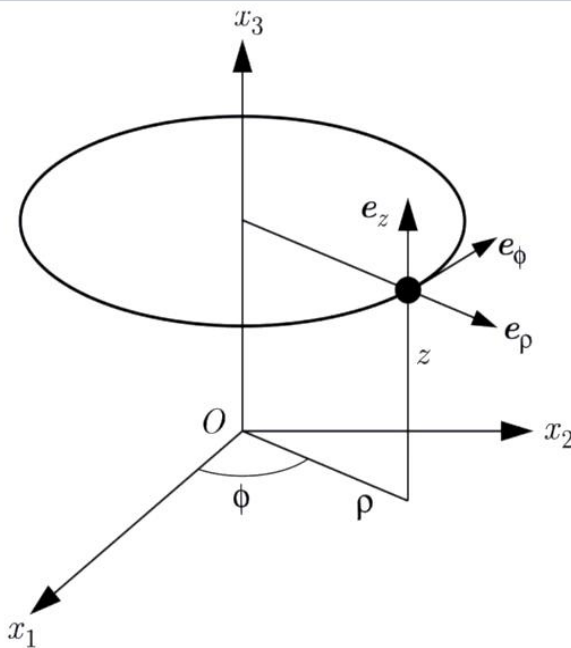
I start with the cylindrical coordinates. Here is the drawing that summarizes our definition of cylindrical coordinates: ρ, ϕ, z , defined with respect to a Cartesian coordinate system presumed to belong to the reference frame. I have the vectors of the reference frame $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$, and I want to calculate the velocity. If I start from the cartesian coordinates of the material point P, which is here. I can do the derivatives but what I will get are the projections onto x_1, x_2, x_3 , of the velocity vector. However, this is not what I want. I would like the projections of the velocity vector onto $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$. You will see that it is very convenient to have these projections. Therefore, I propose to take another approach, the one of first considering the projection of \mathbf{r} on \mathbf{e}_ρ and on \mathbf{e}_z . And now I'm drifting with respect to time. Of course, we'll have a derivative of ρ , with respect to time. We'll have a $\dot{\rho}$ point and then we'll have a \dot{z} point. But I'll show you the final answer. There's one more term. I'll let you pause if you want, to try to see where this term comes from.

Notes

Summary



Dérivée temporelle des vecteurs unité



$$\hat{e}_\rho = \cos \phi \hat{x}_1 + \sin \phi \hat{x}_2$$

$$\hat{e}_\phi = -\sin \phi \hat{x}_1 + \cos \phi \hat{x}_2$$

$$\hat{e}_z = \hat{x}_3$$

$$\frac{d\hat{e}_\rho}{dt} = -\dot{\phi} \sin \phi \hat{x}_1 + \dot{\phi} \cos \phi \hat{x}_2$$

$$\frac{d\hat{e}_\phi}{dt} = -\dot{\phi} \cos \phi \hat{x}_1 - \dot{\phi} \sin \phi \hat{x}_2$$

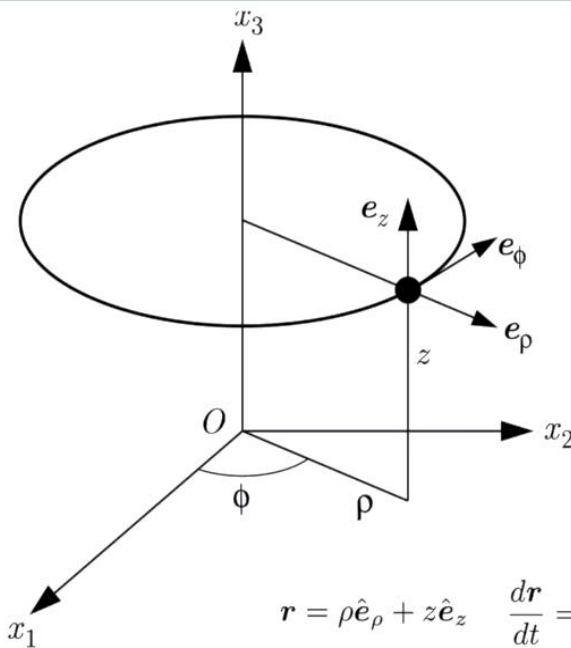
This extra term comes from the fact that when I calculate the derivative of r , with respect to time, I have to take into account that e_ρ depends on time...in general. Indeed, here is the vector e_ρ when ϕ varies in time, e_ρ changes direction. If it changes direction, its derivative is non-zero. So we need to know the derivative of e_ρ with respect to time. I propose to do the calculation in the following way: We had calculated the projections of e_ρ onto x_1 , x_2 , which themselves belong to the frame of reference x_1 , x_2 and x_3 , are independent of time. Then I can calculate the derivative. The derivative of the *cosine* will make *less* ϕ point *sine*. The *sine* ϕ point *cosine*. That's all I need for now but later we'll calculate the acceleration and we'll need the derivative of e_ϕ with respect to time, so I'm doing that, in the run up. Here's $d e_\phi$ over dt . I calculate it. I have a *minus* ϕ point *cos* ϕ minus ϕ point *sin* ϕ appearing here. And now I observe that in $d e_\rho$ on dt , I have the ϕ point with a *less* *sin* ϕ , and *cos* ϕ .

Notes

Summary



Dérivée temporelle des vecteurs unité



$$\hat{e}_\rho = \cos \phi \hat{x}_1 + \sin \phi \hat{x}_2$$

$$\hat{e}_\phi = -\sin \phi \hat{x}_1 + \cos \phi \hat{x}_2$$

$$\hat{e}_z = \hat{x}_3$$

$$\frac{d\hat{e}_\rho}{dt} = -\dot{\phi} \sin \phi \hat{x}_1 + \dot{\phi} \cos \phi \hat{x}_2$$

$$\frac{d\hat{e}_\phi}{dt} = -\dot{\phi} \cos \phi \hat{x}_1 - \dot{\phi} \sin \phi \hat{x}_2$$

$$\frac{d\hat{e}_\rho}{dt} = \dot{\phi} \hat{e}_\phi \quad \frac{d\hat{e}_\phi}{dt} = -\dot{\phi} \hat{e}_\rho$$

$$\mathbf{r} = \rho \hat{e}_\rho + z \hat{e}_z \quad \frac{d\mathbf{r}}{dt} = \dot{\rho} \hat{e}_\rho + \rho \frac{d\hat{e}_\rho}{dt} + \dot{z} \hat{e}_z = \dot{\rho} \hat{e}_\rho + \rho \dot{\phi} \hat{e}_\phi + \dot{z} \hat{e}_z$$

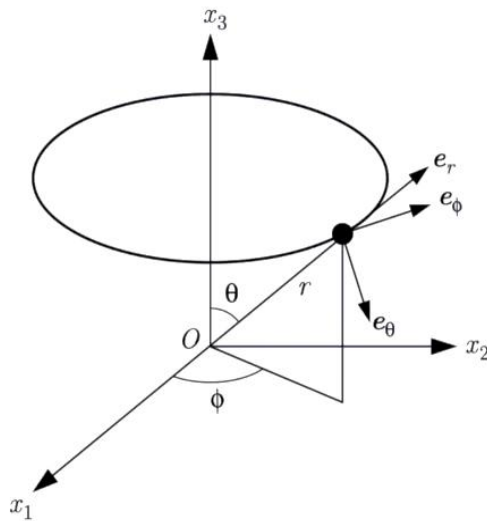
Now $\sin \phi$ and $\cos \phi$, these are the components of \hat{e}_ϕ . I can write $d\hat{e}_\rho$ on dt equals $\dot{\phi}$ times \hat{e}_ϕ . If we look at $d\hat{e}_\phi$ over dt , we have $\cos \phi$ and $\sin \phi$, as here we have $\cos \phi$ and $\sin \phi$. So $d\hat{e}_\phi$ on dt has a $-\dot{\phi}$ point. This $-\dot{\phi}$ point, which appears, times \hat{e}_ρ . There, now we have all the time derivatives of the vectors in our benchmark. Because \hat{e}_z , it doesn't change in time. It always keeps that vertical direction. So I return to the question of calculating the components, on this reference frame, of the velocity vector. I start from \mathbf{r} projected onto this reference frame. I derive with respect to time. I have to introduce this term here: $d\hat{e}_\rho$ on dt , which is here. So I'll have a $\rho \dot{\phi}$ point. That's what I announced...and here's the term.

Notes

Summary



Vitesse projetée sur le repère des c. sphériques



$$x_1 = r \sin \theta \cos \phi$$

$$x_2 = r \sin \theta \sin \phi$$

$$x_3 = r \cos \theta$$

$$\mathbf{r} = r \hat{\mathbf{e}}_r$$

$$\mathbf{v} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\phi} \sin \theta \mathbf{e}_\phi$$

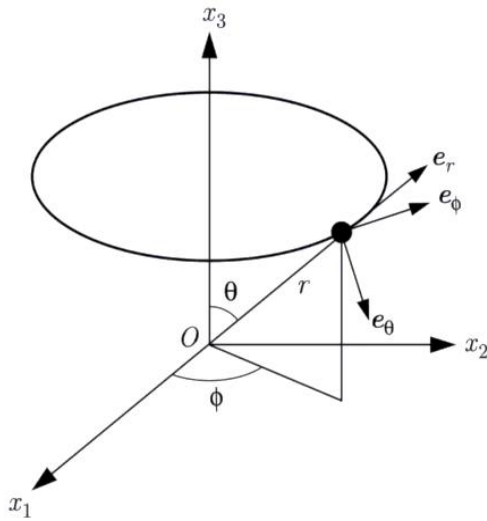
Let's do the same exercise with spherical coordinates. Here are O , x_1 , x_2 , x_3 , belonging to the reference frame. The spherical coordinates r , θ , ϕ ; the associated unit vectors \mathbf{e}_r , \mathbf{e}_θ , \mathbf{e}_ϕ . And now I want to calculate the velocity. I could do this by deriving the Cartesian coordinates with respect to time, but I would again get the projections of the velocity onto x_1 , x_2 , x_3 . But that's not what I want. I want the projections of the velocity onto \mathbf{e}_r , \mathbf{e}_θ , \mathbf{e}_ϕ . Therefore, I propose to start from the projection of the vector \mathbf{r} . So that's the vector there. That's the vector I called r . I'm starting from the projection of r onto the reference frame. It's simply that term there, and I'm going to derive that term with respect to time. The derivative of r will give me a \dot{r} point but \mathbf{e}_r , which is here, changes direction when ϕ varies and when θ varies. \mathbf{e}_r changes direction when θ varies, and also when ϕ varies. So I have some additional terms. I give them here. These are the terms we need to find. We don't have them yet. We need to calculate $d\mathbf{r}$ over dt .

Notes

Summary



Dérivée temporelle des vecteurs unité



$$\frac{d\mathbf{e}_r}{dt} = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi$$

$r \quad r$

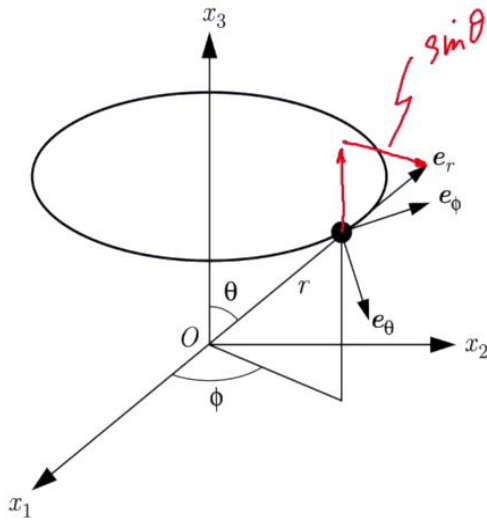
So let's re-examine that question there. Here's our drawing for the spherical coordinates: And now I'm going to do the following: to compute dr over dt . I'm not going to do as for cylindrical coordinates, I'll just write this: by inspection of the figure, I claim I can write this. I suggest you pause and see if you can find the argument by yourself. This is true for the following reason: \mathbf{e}_r , that vector there, changes orientation when θ varies. This means: \mathbf{e}_r keeps a constant norm but undergoes a rotation defined by the angle θ . We saw, when we studied uniform circular motion, We saw this property of vectors that have a constant norm their derivative with respect to time is perpendicular to the vector considered. So we must have a vector perpendicular to \mathbf{e}_r . You can see when θ varies, \mathbf{e}_r goes in that direction there So we are in the direction of \mathbf{e}_θ . So that's what I noted here. And in the same considerations, we had seen that the modulus of the vector was worth the angular velocity. So here, associated with θ , I have a $\dot{\theta}$ point. And we still have to make sure of the sign, but here I think we have the sign the right way around...yes.

Notes

Summary



Dérivée temporelle des vecteurs unité



$$\frac{d\mathbf{e}_r}{dt} = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi$$

$r \quad r \quad v$

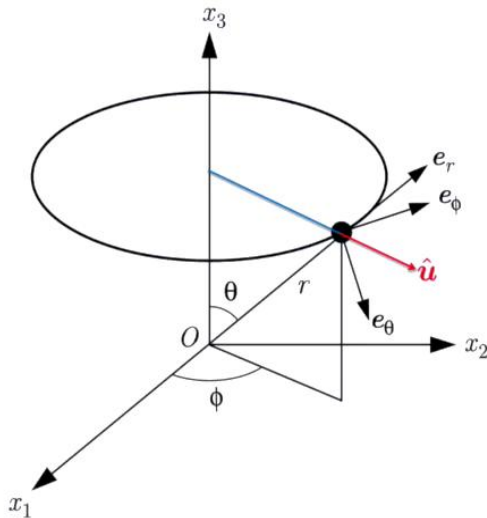
When \mathbf{e}_r goes in that direction there, we go in the direction of the positive \mathbf{e}_θ . But \mathbf{e}_r can also change over time. Its derivative is non-zero because when ϕ varies, \mathbf{e}_r changes orientation. There, we have to do a little auxiliary construction. You have to see that I can write \mathbf{e}_r as the sum of two vectors, one vertical vector, and one horizontal, like this. And it is this horizontal component that undergoes the rotation defined by ϕ . The vertical component, on the other hand, is not affected. What is the length, what is the norm of this vector? Well, since we have here, *one*. The angle θ , we find it here. So here we have $\sin\theta$. So I apply my rule, which we had induced, considering uniform circular motion. I must have a vector worth in norm $\dot{\phi}$ point, the angular velocity, $\dot{\phi}$ point; the norm of the vector $\sin\theta$. And in what direction? So if ϕ varies, \mathbf{e}_r follows along the circle, so we are tangent to the circle, we are in the direction of \mathbf{e}_ϕ . That's what I pointed out here. So here's a geometric argument for why $d\mathbf{r}/dt$ has that shape there.

Notes

Summary



Dérivée temporelle des vecteurs unité



$$\frac{de_r}{dt} = \dot{\theta}e_\theta + \dot{\phi}\sin\theta e_\phi$$

$$\frac{de_\phi}{dt} = -\dot{\phi}\hat{u}$$

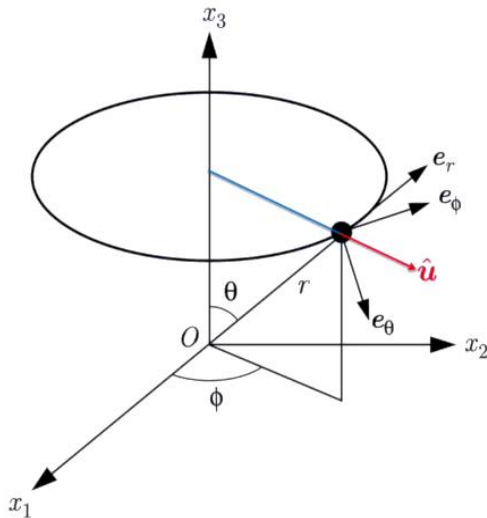
This is the only time in the course, when we need to engage in this exercise there. It's not a technique I ask you to learn, but it's important to me, that you see where these formulas come from. And I chose the method that required a minimum of algebra. I now turn to $d e_\phi$ over dt . We don't need it to calculate the speed. But later on, we're going to need to calculate the vector acceleration. We'll need this term. So I'm doing that calculation now. $d e_\phi$ over dt . There's e_ϕ . So e_ϕ , it's easier. e_ϕ does not change direction as r varies. And e_ϕ does not change direction when θ varies. e_ϕ changes direction only when ϕ varies. The derivative of e_ϕ with respect to time, must be perpendicular to e_ϕ , in the horizontal plane. So in the direction of this vector u that I drew here. So there you have it. I have to say that my derivative $d e_\phi$ on dt is in the u direction. I need to put the angular velocity, the norm of the vector, which is $u\dot{\phi}$, and the sign, by inspection of the graph. I see that as ϕ increases e_ϕ goes in that direction there, so opposite to u .

Notes

Summary



Dérivée temporelle des vecteurs unité



$$\frac{de_r}{dt} = \dot{\theta}e_\theta + \dot{\phi}\sin\theta e_\phi$$

$$\frac{de_\phi}{dt} = -\dot{\phi}\hat{u}$$

$$\hat{u} = \sin\theta e_r + \cos\theta e_\theta$$

$$\frac{de_\theta}{dt} = -\dot{\theta}e_r + \dot{\phi}\cos\theta e_\phi$$

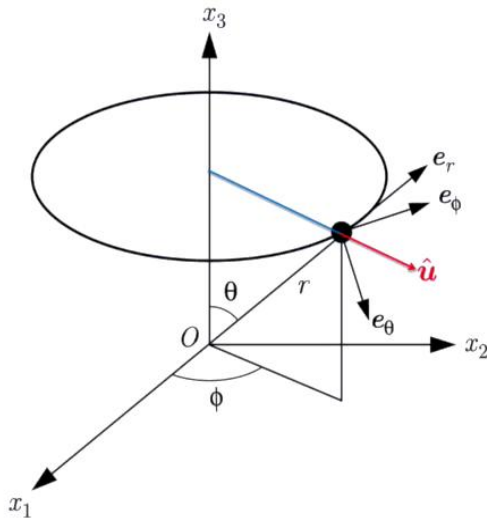
That's why I put this *less* sign here. Now, u , we still have to project u onto e_r and e_θ . Then we see that if this line there, which defines the angle θ , is perpendicular to that one... and this line is perpendicular to that one. I remind you that e_θ is tangent to the circle. Of coordinates θ varies, so we have a right angle, here, and with what I just said, so we have the angle θ which ends up there. So u has a projection ... u has a projection $\cos\theta$ onto e_θ . That's what I noted here. And we have a $\sin\theta$ with the sign *plus*, in the perpendicular direction, so the direction of e_r . There, with that we are done with $d e_\phi$ on dt . We are left with $d e_\theta$ on dt . Then we'll use the same argument as for $d e_r$ on dt . e_θ is here. When θ varies, e_θ rotates entirely. So, we must write: $\dot{\theta}$ point. We put a *minus* sign because we observe that as θ increases e_θ rotates in that direction. So opposite to e_r . And it is indeed in the direction of e_r . Everything happens in the vertical plane that contains x_3 , OP , that vertical, and that segment there.

Notes

Summary



Dérivée temporelle des vecteurs unité



$$\frac{d\mathbf{e}_r}{dt} = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi$$

$$\frac{d\mathbf{e}_\phi}{dt} = -\dot{\phi}\hat{\mathbf{u}}$$

$$\hat{\mathbf{u}} = \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta$$

$$\frac{d\mathbf{e}_\theta}{dt} = -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi$$

$$\frac{d}{dt}\mathbf{r} = \dot{r}\hat{\mathbf{e}}_r + r\frac{d}{dt}\hat{\mathbf{e}}_r$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\phi}\sin\theta\mathbf{e}_\phi$$

So we're perpendicular to \mathbf{e}_ϕ . There is no component along \mathbf{e}_ϕ , only along \mathbf{e}_r . \mathbf{e}_θ also varies when ϕ varies. \mathbf{e}_θ starts to rotate, like the material point on the circle. And therefore, we have a derivative in the direction of \mathbf{e}_ϕ . That's what I noted here. There's only the horizontal component, so...that one, that comes into play. And that we had seen that, you remember, We had made the argument, like, there was the angle θ . So it's a $\cos\theta$ that comes into play. And the angular velocity is $\dot{\phi}$ point. There! I got the three derivatives of the three vectors in the spherical coordinate frame. I can now move on to the question of calculating the velocity. r is simply r in the \mathbf{e}_r direction. When I drift, I have that term there. And then I have the derivative of \mathbf{e}_r , with respect to time. Which is here. So, we'll have a $r\dot{\theta}$ point, in the \mathbf{e}_θ direction. $r\dot{\phi}\sin\theta$, in the \mathbf{e}_ϕ direction. This is what I had announced. Here, I repeat, is the formula.

Notes

Summary





Coordonnées cylindriques :

$$\mathbf{r} = \rho \hat{\mathbf{e}}_\rho + z \hat{\mathbf{e}}_z$$

$$\mathbf{v} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\phi} \mathbf{e}_\phi + \dot{z} \mathbf{e}_z$$

Coordonnées sphériques :

$$\mathbf{r} = r \hat{\mathbf{e}}_r$$

$$\mathbf{v} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\phi} \sin \theta \mathbf{e}_\phi$$

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I summarize. For the cylindrical coordinates, here is the vector \mathbf{r} , so the vector of *position* projected onto the reference frame associated with the cylindrical coordinates. Here is the velocity, expressed with the ρ , ϕ , z coordinates. And projected onto the reference frame of the cylindrical coordinates. Likewise for the spherical coordinates. Here's the *position* vector, projected onto the spherical coordinate frame. And here is the velocity we just obtained, projected onto the spherical coordinate frame.

Notes

Summary



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