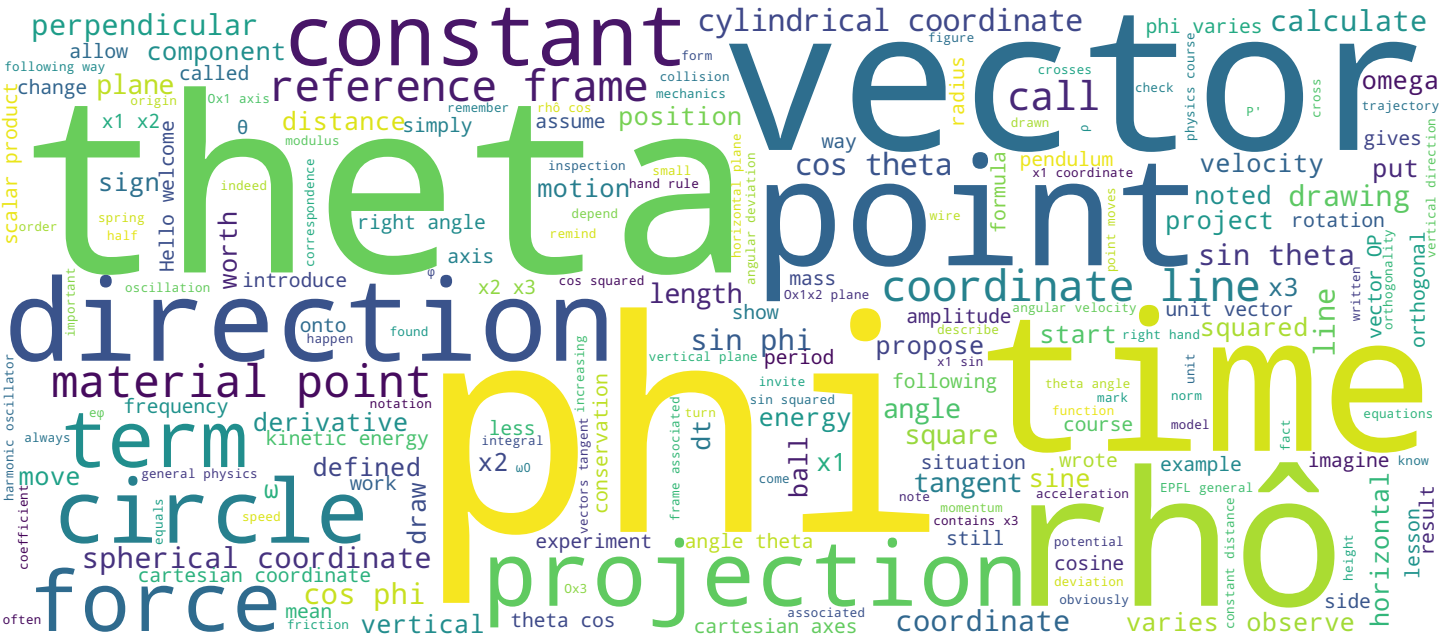
Coordonnées cylindriques et sphériques

Mécanique, cours 7.1

Jean-Philippe Ansermet

http://go.epfl.ch/traite-meca-1-7



Video

Coordonnées cylindriques et sphériques

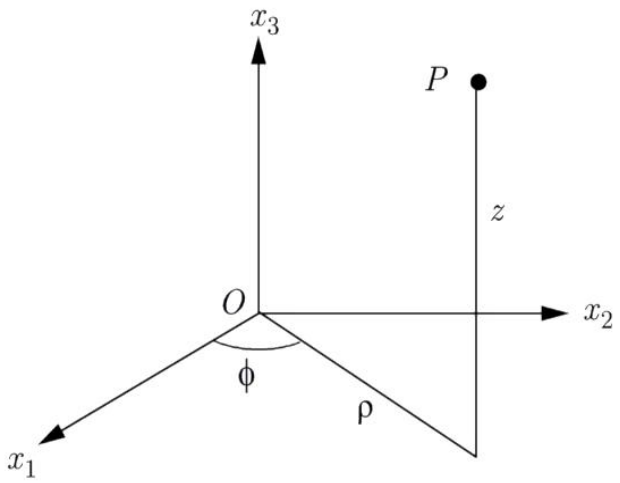
Hello. Welcome to the EPFL general physics course. In this lesson, I will introduce cylindrical and spherical coordinates. Until now, we have done problems that we could treat properly with cartesian coordinates, but in mechanics we often find situations with particular symmetries and it is important to use a system of coordinates that allow to account for these symmetries. That's why we're going to see the cylindrical coordinates, the spherical coordinates and for these two systems of coordinates, I'm going to define landmarks that are associated with them.

Notes

Summary

0m 04s

Définition : coordonnées cylindriques



$$P(\rho, \phi, z)$$

$$x_1 = \rho \cos \phi$$
$$x_2 = \rho \sin \phi$$
$$x_3 = z$$

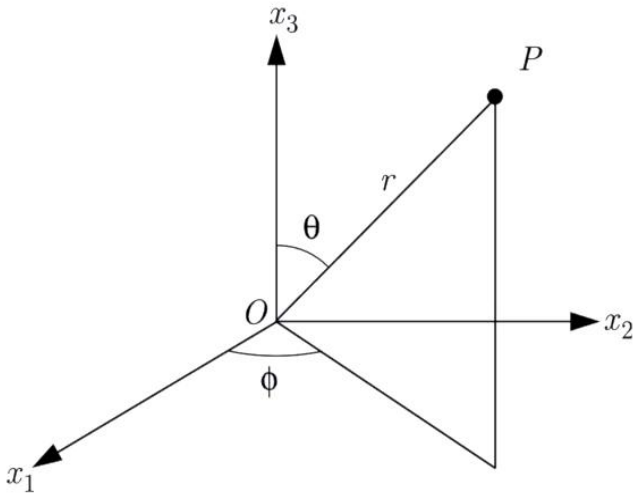
I'll start with the cylindrical coordinates. I imagine that my reference frame is materialized by a system of cartesian axes O, x1, x2, x3. I want to locate the position of my material point, until now, we have defined the position of the material point with its cartesian coordinates, but now I propose the cylindrical coordinates which are defined in the following way: first, I will consider the projection *P'* of the material point*P* on the plane *Ox1x2* and this height above the plane I will call itz. *z* is equivalent to *x3* but I'll use the notation *z* to signal that we're in cylindrical coordinates. Then I'll use *p*, the distance from *P* to the *Ox3* axis, so it's that length there. And finally, I'm going to use the angle that marks the angular deviation between the Ox1 axis and that axis corresponding to the projection of P on the Ox1x2 plane. So now, my material point has its position which is defined by rhô, phi and z. Of course we can try to write the correspondence between the cartesian coordinates of the point and the cylindrical coordinates.

Notes

Summary

0m 48s

Définition : coordonnées sphériques



$$P\,(r, \theta, \phi)$$

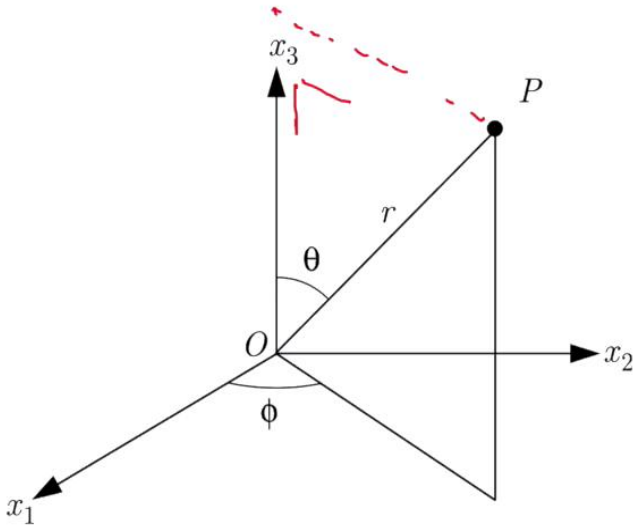
So, if we look at the drawing, the x1 coordinate is going to be given by the projection of this point *P'* which will be about here, here I have a right angle and so I have a right triangle with the right angle here, this length there, the x1 coordinate, is rhô cos phi. Which I wrote like this. The x2 coordinate is this distance here, it's rhô sin phi and z is simply equal to x3. Now I'll move on to the definition of the spherical coordinates. Once again, I give myself a system of Cartesian axes linked to the reference frame and I want to characterize the position of the material point P. Now I do it in the following way: it is in the logic of spherical coordinates, I give myself the distance of P from the origin, r, and I give myself the angle theta characterizing the deviation of the vector OP from Ox3, theta is the deviation of OP from the axis Ox3, and finally, as for the cylindrical coordinates, I consider the projection of P in the Ox1x2 plane, the angular deviation of OP' with respect to the Ox1 axis is given by the angle phi as for the cylindrical coordinates. Now my material point is characterized by the coordinates r, theta, phi. r, theta and phi.

Notes

Summary

2m 39s

Définition : coordonnées sphériques



$$P\,(r, \theta, \phi)$$

$$x_1 = r \sin\theta \cos\phi$$
$$x_2 = r \sin\theta \sin\phi$$
$$x_3 = r \cos\theta$$

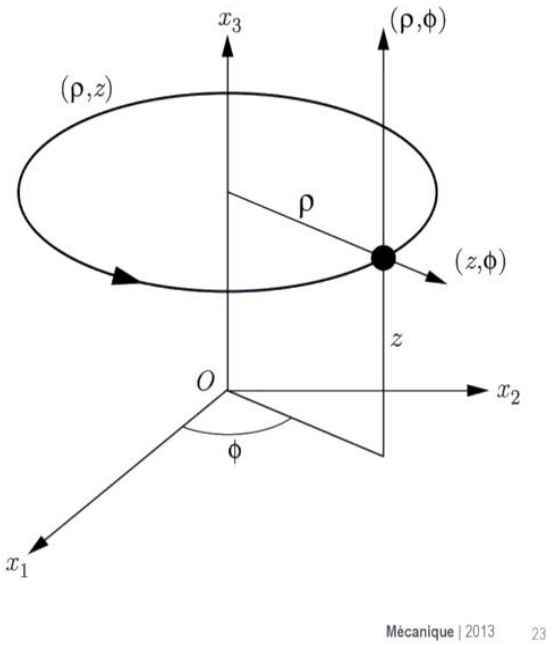
Once again we can look for the correspondence between the cartesian coordinates of point P and the spherical coordinates of point P. Then we have to see the following thing: this distance is worth, I can make an auxiliary drawing like this, I make the projection of P on the Ox3 axis, here I have a right angle and so I have this distance which is worth r cos theta in the x3 direction, that's what I noted here, this is equal to r sin theta which is equal to the length of this segment, and now this r sin theta must be projected on x1 and on x2, so we will have a cos phi for the projection on x1, sin phi for the projection on x2, that's what I have here, so I have my r sin theta with the projection cos phi on x1, sin phi on x2.

Notes

Summary

4m 48s

Définition : lignes de coordonnées (c. cylindriques)



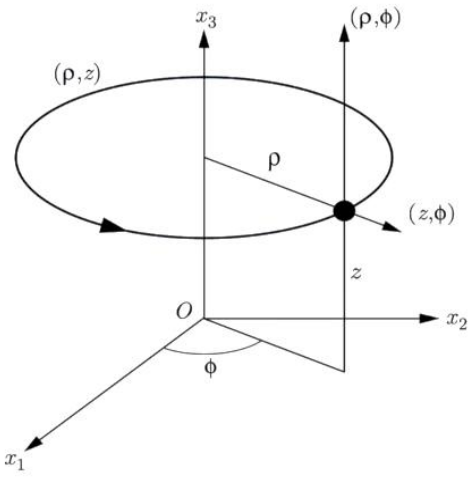
To define the landmarks associated with these coordinate systems I need to define coordinate lines. Let's first look at the case of cylindrical coordinates. So on this drawing we have rhô which is here, phi there, z, the third coordinate for the cylindrical coordinates. Now if phi varies, but rhô is constant and z is constant, z constant, we are in a plane parallel to the Ox1x2 plane, let's call this horizontal plane for the sake of convenience, so we're going to be in a horizontal plane at height z and we're at a constant distance rhô from the x3 axis, so we're on the circle I've drawn here, this circle is called a coordinate line, it's the coordinate line when only phi varies, I'm going to write rhô, z to indicate the two coordinates that remain constant on this line. Now, if I vary rhô, I simply have the material point moving on this line, so here is another coordinate line, the coordinate line where z and phi are constant. Finally, I have the line of coordinates where only z varies and the material point moves on this vertical with rhô and phi remaining constant.

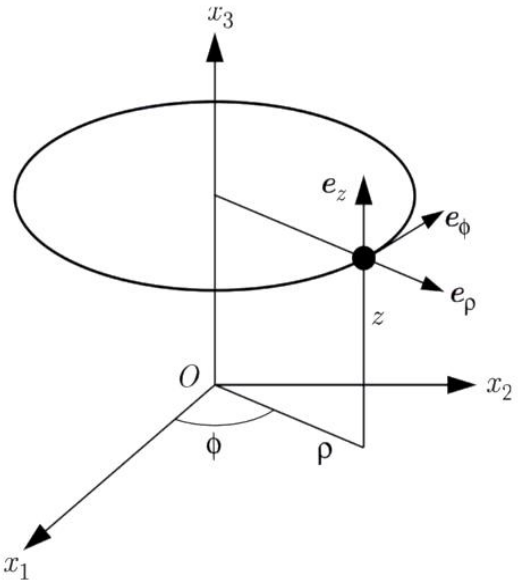
Notes

Summary

5m 58s

Définition : repère associé (c. cylindriques)





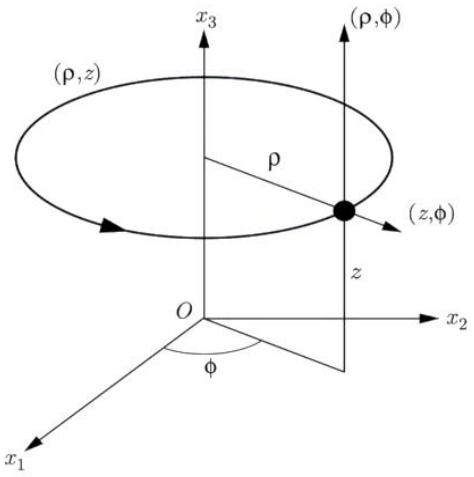
Now I can define the reference frame associated with the cylindrical coordinates. I take back my drawing, this drawing sums up the situation well, a reference frame materialized by Ox1x2x3, the coordinates rhô, phi, z, here are the coordinate lines and now I propose to define vectors of length one, unit vectors tangent to the coordinate lines. So a coordinate line, the rhô coordinate line varies, simply I would have a vector in that sense, which I will call e rhô, the vector tangent to the coordinate line rhô alone varies, we obviously have a vector e z when only z varies, and if phi varies, we want a unit vector tangent to the coordinate line, so it will be in that direction and I will always agree to take my vectors in the direction corresponding to the increasing coordinate, rhô increasing in that direction, z increasing in that direction, phi increasing in that direction, so I define an e phi like this. I'm drawing a clean picture. Here it is with e rhô, e phi, e z, I agreed that the vectors had length one, I observe that e z is vertical, e rhô and e phi are horizontal, so they have orthogonal, e phi is tangent to the circle, so e phi is perpendicular to that radius and therefore, e phi is perpendicular to e rhô.

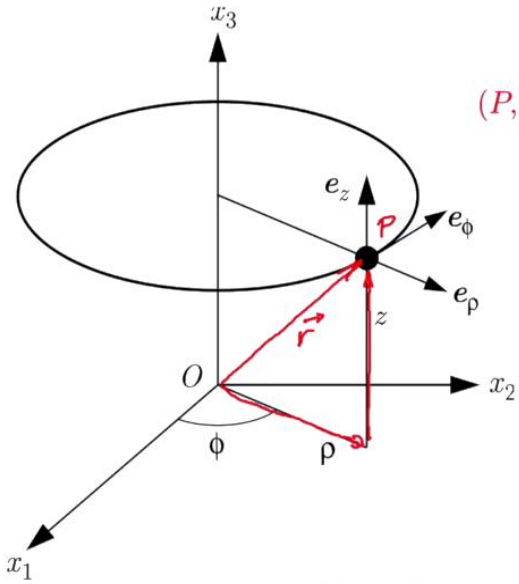
Notes

Summary

7m 39s

Définition : repère associé (c. cylindriques)





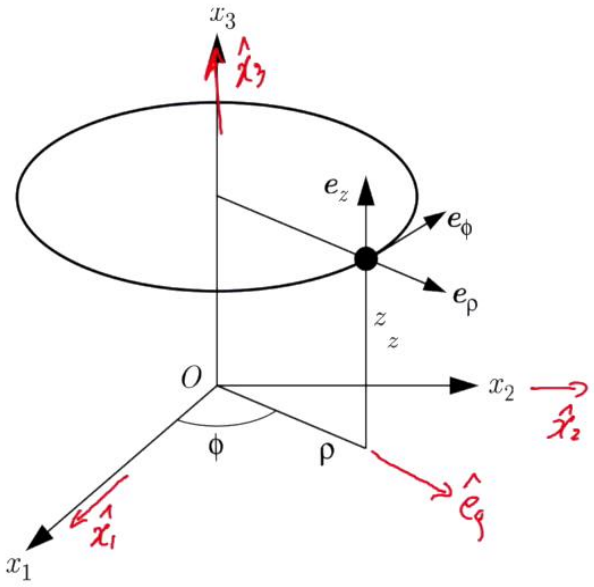
So we have three orthogonal, normed vectors, moreover, if now I agree to call the reference frame associated with the point P e rhô, e phi, e z, in this order, we have e rhô cos e phi, right hand rule, e rhô cross e phi is in the direction of e z, so we have a direct orthonormal reference frame. Now I'm going to note the vector OP, this is our point P, here is the vector OP which I call vector r, I want to project the vector r on my reference frame e rhô, e phi, e z. I observe that the vector r I can draw as a sum of two vectors, this vector and that vector, this one is rhô in the direction e rhô and that one is z in the direction e z, that's what I wrote here, the vector r has two components, a component according to e rhô, a component according to e z.

Notes

Summary

9m 40s

Coordonnées cartésiennes des vecteurs du repère



$$\hat{e}_\rho = \cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2$$

$$\hat{e}_\phi = -\sin\phi\,\hat{x}_1 + \cos\phi\,\hat{x}_2$$

$$\hat{e}_z = \hat{x}_3$$

$$\hat{e}_\rho \cdot \hat{e}_\phi = 0$$
$$\hat{e}_\rho \cdot \hat{e}_z = 0$$
$$\hat{e}_\phi \cdot \hat{e}_z = 0$$

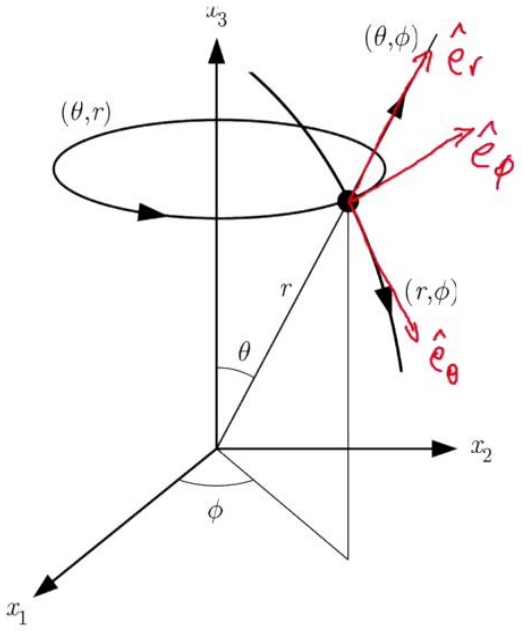
Now we can be interested in calculating we'll need it later, we're going to calculate the projections of e rhô, e phi, e z on the cartesian axes x1, x2, x3, so I'm assuming that I've defined vectors x1 hat, x2 hat, an x3 hat which I find here, and now I propose to calculate the projection for example of e rhô. I can draw it again here, e rhô, at this point we can see how the projections of e rhô will be constructed, we will have a cos phi in the x1 direction and a sin phi in the x2 direction, this is what I have noted here, cos phi, sin phi. e phi by inspection of the drawing I grant you, it's a little difficult to see what the projections are but we remember that e phi must be perpendicular to e rhô, the scalar product, so this component times this one, plus this one times that one must be zero. So we must cross the sines and cosines and we must introduce a minus sign. We observe that e phi points to the side opposite x1, e phi is opposite x1, so I put the minus sine, I always arrange to make drawings with acute angles, which allows me to see the signs correctly by inspection of the drawings. e z is simply identical to x3. Now I see that e z is perpendicular to the other two vectors and that e rhô and e phi, so by construction, are perpendicular, so I have all these results.

Notes

Summary

11m 03s

Définition : repère associé (c. sphériques)



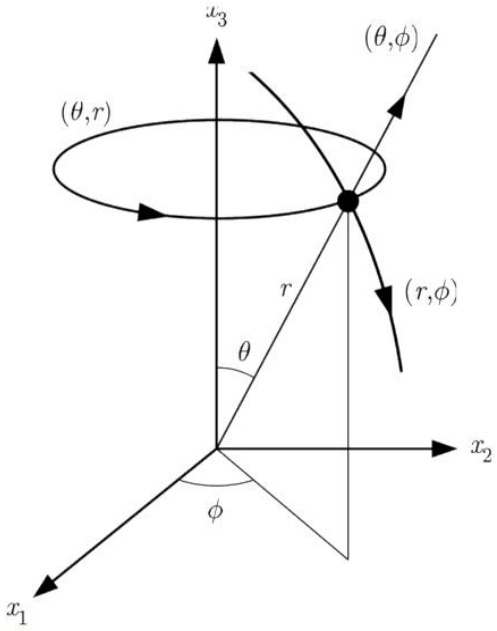
I now turn to the coordinate lines for the spherical coordinates. Here is a drawing that summarizes the definition of spherical coordinates. r, theta, phi. If only r varies, the material point moves on this line, so I have the coordinate line where theta and phi are constant. We see moreover how the two angles theta and phi define a direction of space. If theta is constant and r is constant, constant theta, the material point must move on a cone, at a fixed distance from r, we are on a circle, we are on this circle. Here is the circle where theta is constant and r is constant, only phi varies. Finally, I have to draw the coordinate line when r is constant, phi is constant, but theta varies. Constant phi means that we are in the plane which contains this line, this one, that one and that one, we are in this plane, at a constant distance r from the origin we describe a circle and this circle has more or less this shape, here is the circle of theta only varies. I take this drawing and now I will draw the unit vectors tangent to the coordinate lines. The simple e r is there, e theta is tangent to the circle, there's my e theta, and the e phi is tangent to the other circle, harder to see how it's oriented, about like that, there's e phi.

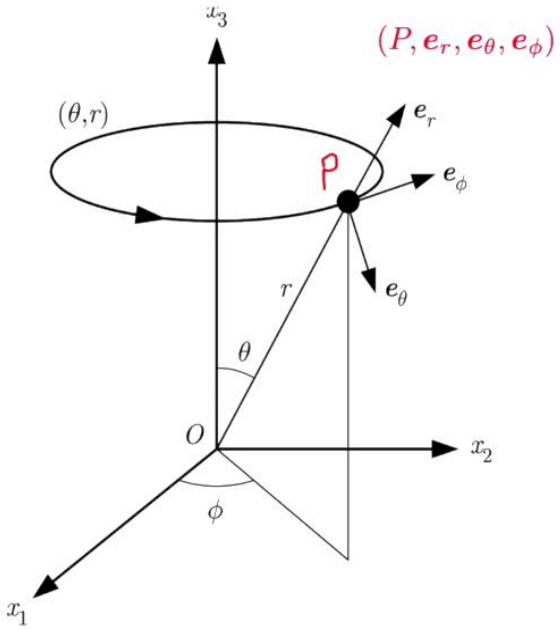
Notes

Summary

13m 08s

Définition : repère associé (c. sphériques)





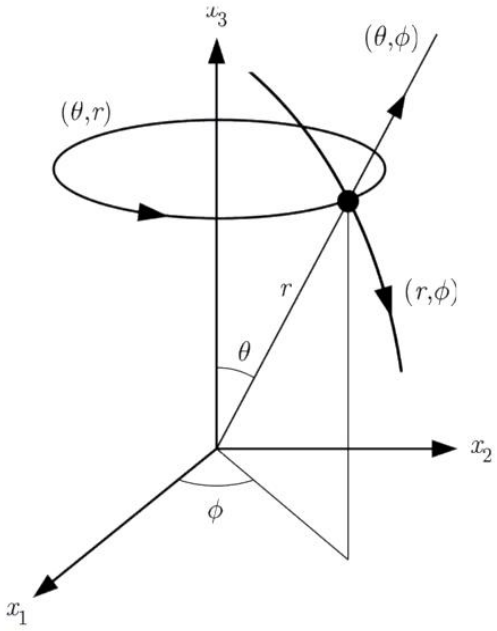
I take vectors of norm one, let's see the drawing more neatly, here's e r, e theta, e phi, I have, not to burden the figure, I've removed this arc of a circle but you have to remember that the e theta is tangent to a circle in the plane that contains x3, the radius vector and this line. I can define my reference frame related to the point P, it's always that point it's the point P, the reference frame follows the point P, in P I have the vectors e r, e theta, e phi taken in that order so that e r crosses e theta gives e phi, so I have by the right hand rule, e r crosses e theta goes in the direction of e phi, so I have here a direct reference frame, we still have to be sure of the orthogonality of all the vectors. Then, the vector e theta being tangent to this circle, it is orthogonal to this radius, so e theta is perpendicular to e r. Now e phi is tangent to a circle, let's call this plane Ox1x2 the horizontal again, this ellipse is horizontal, e phi is horizontal, but e theta and e r are in the plane which contains x3, OP and this vertical, it's a vertical plane so e r and e theta are in a vertical plane, while e phi is in a horizontal plane so they are orthogonal.

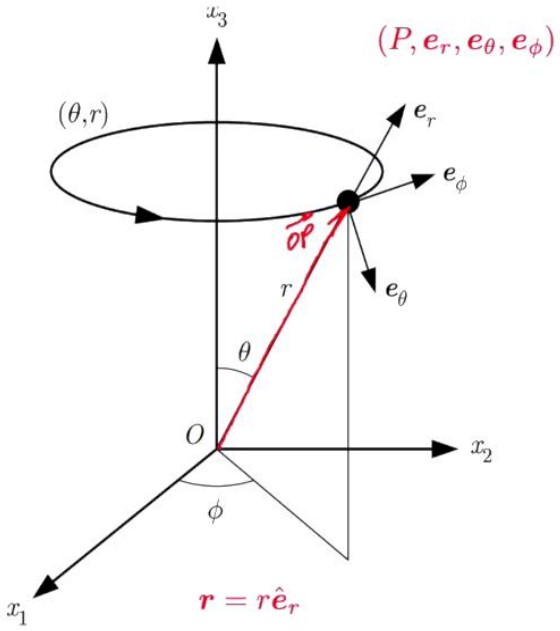
Notes

Summary

15m 11s

Définition : repère associé (c. sphériques)





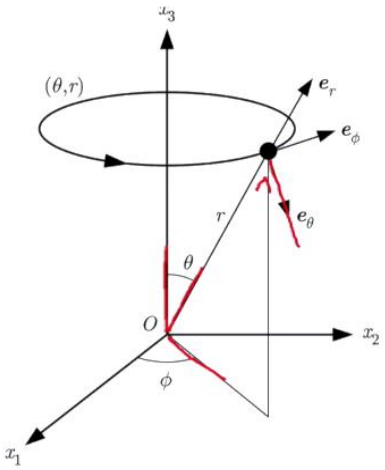
There we have the orthogonality of our three vectors. I'm going to need the projection of the vector r, so this is the vector OP as usual here's my vector I could call OP, which I call to simplify r, r I'm going to project it onto my reference frame and I just have a component along e r. So there, there are often difficulties, students tend to want to introduce other terms, there is only this term, the projection of r, r is in fact along e r so we just have this formula.

Notes

Summary

17m 13s

Coordonnées cartésiennes des vecteurs du repère



$$\hat{\boldsymbol{e}}_r = \sin\theta\cos\phi\hat{\boldsymbol{x}}_1 + \sin\theta\sin\phi\hat{\boldsymbol{x}}_2 + \cos\theta\hat{\boldsymbol{x}}_3$$

$$\hat{\boldsymbol{e}}_\theta = \cos\theta\cos\phi\hat{\boldsymbol{x}}_1 + \cos\theta\sin\phi\hat{\boldsymbol{x}}_2 - \sin\theta\hat{\boldsymbol{x}}_3$$

$$\hat{\boldsymbol{e}}_\phi = -\sin\phi\hat{\boldsymbol{x}}_1 + \cos\phi\hat{\boldsymbol{x}}_2$$

$$\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_\theta = 0$$
$$\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_\phi = 0$$
$$\hat{\boldsymbol{e}}_\phi \cdot \hat{\boldsymbol{e}}_\theta = 0$$

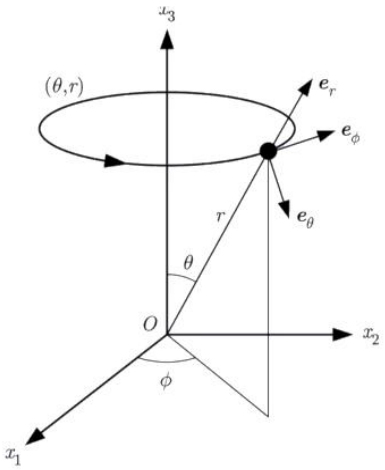
We can have fun as an exercise projecting e r, e theta, e phi onto the x1, x2, x3 Cartesian axes, so let's go ahead and try to do that. I start with e r. I draw an auxiliary line here vertical parallel to x3 and then a horizontal line like this. The component of e r in the vertical is this term which is cos theta because the theta angle I find here, the theta angle it's there, so I have a cos theta in the vertical direction, that's what I noted here, in the horizontal I have this length which is sin theta, this sin theta if you want you can draw it here in the extension and you have to project it on x1 and on x2, which will give you a cos phi and a sin phi, that's what I have noted here, you have the cos phi and the sin phi for the sin theta term. Now I'm going to look at the projection of e theta. So, for e theta, you have to try to figure it out. One way to do this is to realize that what defines theta is this angle here, and this side of the angle theta is perpendicular to this one and this side of the angle theta is perpendicular to this one. I remind you that e theta is tangent to the circle of coordinates theta alone varies and so e theta is perpendicular to the vector ray at a right angle here.

Notes

Summary

18m 00s

Coordonnées cartésiennes des vecteurs du repère



$$\hat{\boldsymbol{e}}_r = \sin\theta\cos\phi\hat{\boldsymbol{x}}_1 + \sin\theta\sin\phi\hat{\boldsymbol{x}}_2 + \cos\theta\hat{\boldsymbol{x}}_3$$

$$\hat{\boldsymbol{e}}_\theta = \cos\theta\cos\phi\hat{\boldsymbol{x}}_1 + \cos\theta\sin\phi\hat{\boldsymbol{x}}_2 - \sin\theta\hat{\boldsymbol{x}}_3$$

$$\hat{\boldsymbol{e}}_\phi = -\sin\phi\hat{\boldsymbol{x}}_1 + \cos\phi\hat{\boldsymbol{x}}_2$$

$$\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_\theta = 0$$
$$\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_\phi = 0$$
$$\hat{\boldsymbol{e}}_\phi \cdot \hat{\boldsymbol{e}}_\theta = 0$$

I extend this line and so, if here I have theta, there I have again the angle theta. Therefore, I have a cos theta in this direction, if you want I can mark the projection here, this length here will be cos theta and this cos theta I have to project on x1 and on x2, which will give me a cos phi and a sin phi. Let's see, I indeed have cos theta appearing and I have cos phi and sin phi as expected. In the vertical direction, the e theta has a projection which will be cosine of the angle complementary to theta, so sine of theta. There is a minus sign because we can see that theta is directed in the opposite direction of x3. Remain e phi. For e phi, there is nothing new compared to the projection of the vector e phi that we had calculated for the cylindrical coordinates, so we have minus sin phi and cos phi. Having written the components of e r, e theta and e phi on x1, x2, x3, we still have plenty of time to check these orthogonalities, let's take for example e r and e theta, here I have cos theta sin theta with a minus sign, here I have sin theta cos theta with sin squared, here I have sin theta cos theta with cos squared, cos squared plus sin squared makes one, I have more sin theta cos theta left, here I have less sin theta cos theta, so I have my zero. You can also check the other two scalar products, you will get again the zero.

Notes

Summary

20m 02s