ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Pendule paramétrique

http://go.epfl.ch/traite-meca-5-30

Mécanique, cours 28.2

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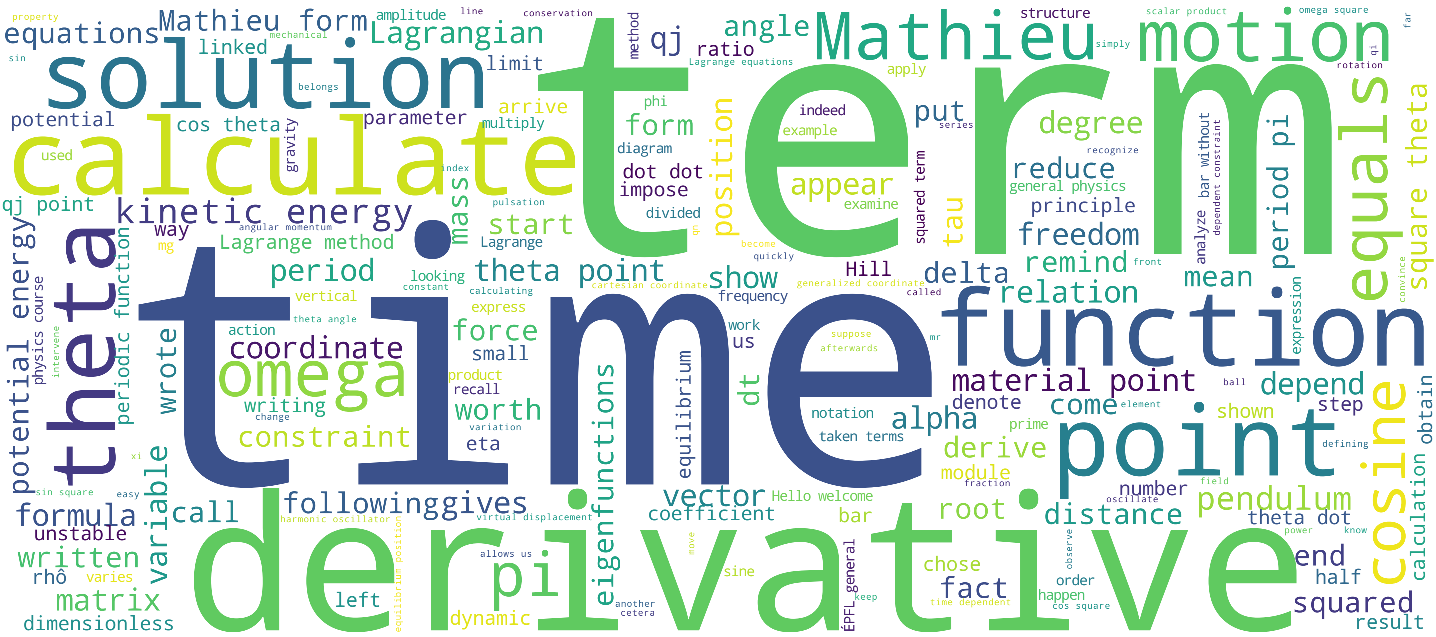

Video

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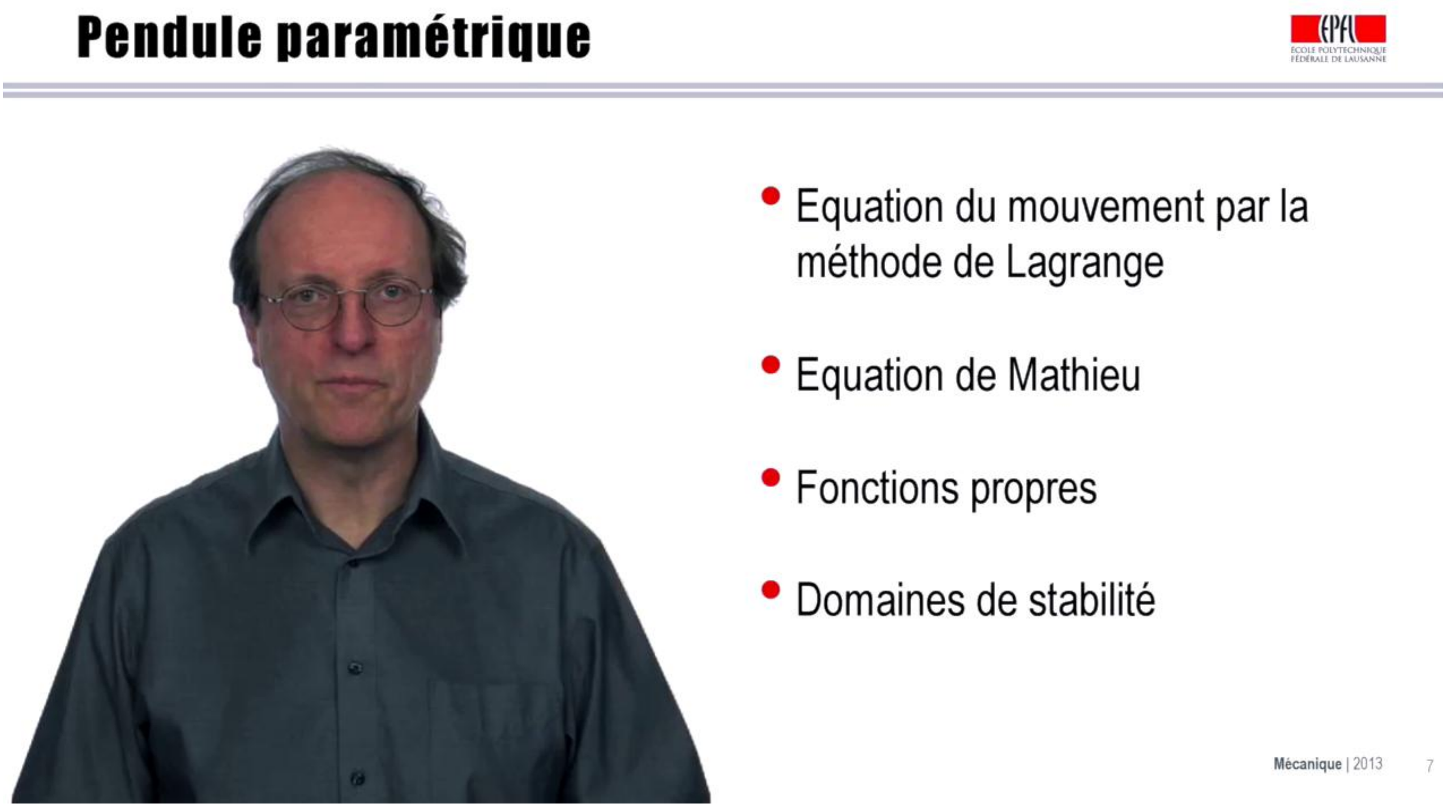


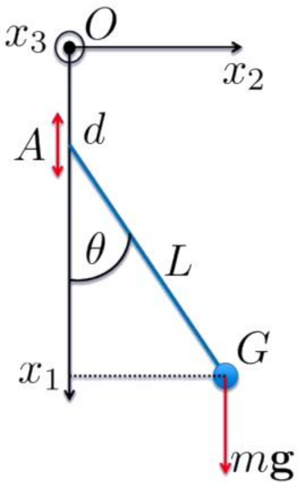
Hello, welcome to the ÉPFL general physics course. In this module, we will analyze the dynamics of a pendulum, undergoing a parametric excitation I will start by defining the problem, we will write the equations of motion for this pendulum using the Lagrange method, then we will see how this equation has the form of Mathieu equation, we will look for the eigenfunctions, and we will discuss the stability domains of this pendulum.

Notes

Summary

0m 04s

Pendule forcé : méthode de Lagrange



$$x_1 = d + L\cos\theta \qquad x_2 = L\sin\theta$$

$$\dot{x}_1 = \dot{d} - L\dot{\theta}\sin\theta \qquad \dot{x}_2 = L\dot{\theta}\cos\theta$$

$$T = \frac{1}{2}m\left(\dot{d}^2 - 2\dot{d}L\dot{\theta}\sin\theta + L^2\dot{\theta}^2\right)$$

$$V = -mg\left(d + L\cos\theta\right)$$

$$L = \frac{1}{2}m\dot{d}^2 - m\dot{d}L\dot{\theta}\sin\theta + \frac{1}{2}mL^2\dot{\theta}^2 + mgd + mgL\cos\theta$$

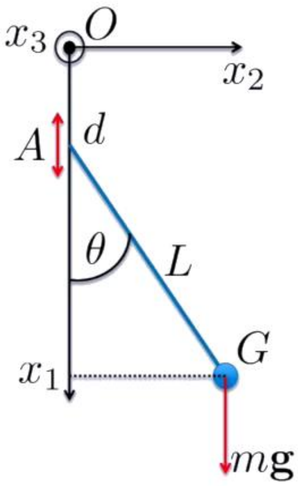
Here is the mechanical system, you have a rigid bar without mass, a mass m at the end, in the field of gravity. The end of the bar at point A oscillates along a vertical, and I denote by the distance between point O which belongs to the differential and point A, I denote this distance by d, d is here a function of time. I will use Lagrange's method to obtain the equation of motion. This is a one degree of freedom problem, and I will use the theta angle as the coordinate; d varies, and d which depends on time is a time dependent constraint. So we have one degree of freedom. To calculate the kinetic energy, I will start by calculating the position, so I write it in cartesian coordinates, using my theta coordinate, I calculate the derivative, beware that here d, this d, is a function of time, so d point non-zero, the other derivatives are trivial, I calculate the kinetic energy, you have this squared term, this squared term is combined here we have a sin square theta, there a cos square theta, so that gives this term, and there is the double product here, that I wrote, here it is. The potential energy is this distance plus this one downwards, so it's minus mg with a d plus L cos theta, the lagrangian is T minus V, here it is, I've written minus V here, and now we have to calculate the derivatives.

Notes

Summary

0m 42s

Pendule forcé : équation du mouvement



$$L = \frac{1}{2}m\dot{d}^2 - m\dot{d}L\dot{\theta}\sin\theta + \frac{1}{2}mL^2\dot{\theta}^2 + mgd + mgL\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = -m\dot{d}L\sin\theta + mL^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -m\ddot{d}L\sin\theta - m\dot{d}\dot{\theta}\cos\theta + mL^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -m\dot{d}L\dot{\theta}\cos\theta - mgL\sin\theta$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta - \frac{\ddot{d}}{L}\sin\theta = 0$$

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I'm rewriting the Lagrangian to make the calculation more convenient, I have to calculate d of L over d of theta point, so there's a term in theta point here, and there. This is a derivative with respect to theta point taken as a variable. This term here. I have to I derive with respect to time, which, in principle, depends on time, so I have a d dot dot, the theta sign gives a contribution too, and the theta dot gives a theta dot dot. d of L on d of theta, it has a term that comes from here, from a term of kinetic energy, and another one of potential energy. Here it is. So we have the equation of motion, I've simplified here by, I've divided the equation by L square, here's the equation of motion for this pendulum at any angle.

Notes

Summary

2m 38s

Equation de Hill

$$\ddot{\theta} + \frac{g}{L}\sin\theta - \frac{\ddot{d}}{L}\sin\theta = 0$$

$$d = d_0\cos(2\Omega t)$$

$$\theta \to 0$$

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{4d_0\Omega^2}{L}\cos(2\Omega t)\right)\theta = 0$$

$$\ddot{x} + G(t)x = 0 \qquad x = L\theta$$

$$\text{période : } \tau = \frac{\pi}{\Omega}$$

Now I'll show you how this problem reduces to a Mathieu equation, first we'll convince ourselves that we have a Hill equation, to do so, we'll look at the limit of small angles, and then we'll impose so that d has a periodic function. I chose, to be able to easily reduce to Mathieu's equation, I chose one, to write two omega t, rather than omega t, and I will keep the two, like in Mathieu's equation. If theta tends to zero, we have the sign theta here, which becomes theta, and so we have these two terms, and here we have an equation which has the structure of a Hill equation, with here an oscillating term, so this function is well a periodic function, I can multiply theta by L if I want to have an x, which has the form of a Hill function, and now I'm going to treat these parameters to have a dimensionless equation, it will be for Mathieu's equation, I remind you that the period, since there is a two here, the period is pi over omega, in this problem.

Notes

Summary

3m 36s

Equation de Mathieu

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{4 d_0 \Omega^2}{L} \cos(2\Omega t) \right) \theta = 0$$

$\bar{t} = \Omega t$

$\Omega^2 \dfrac{d^2\theta}{d(\Omega t)^2}$

θ''

$$y'' + (p - 2q \cos 2\bar{t}\,)\, y = 0$$

Now we'll see how this equation reduces to the Mathieu form, here's the equation, here's the Mathieu form. So we see that here, we should be able to write the second derivative with respect to time in this way. We write omega t squared, by putting omega squared in front of it, this term is dimensionless, so we'll call the time t bar omega t, and here we have the theta second derivative.

Notes

Summary

4m 59s

Equation de Mathieu

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{4d_0\Omega^2}{L}\cos(2\Omega t)\right)\theta = 0$$

$$y'' + (p - 2q\cos 2\bar{t}\,)\,y = 0$$

$$\bar{t} = \Omega t$$

$$p = \frac{g}{L\Omega^2} \qquad q = \frac{-2d_0}{L}$$

We have Mathieu's form, with t bar without unit, and p and q, which are the ratios, the g ratios on L that I have here, and then this ratio divided by the omega square, which comes from the second derivative with respect to time.

Notes

Summary

5m 46s

$$y'' + (p - 2q\cos 2\bar{t}\,)\,y = 0$$

chercher une solution de période π :

$$\bar{e}_1(\bar{t}) = \sum_{r=0}^{\infty} A_{2r}\cos 2r\bar{t}$$

$$2\cos 2r\bar{t}\cos 2\bar{t} = \cos(2r+2)\bar{t} + \cos(2r-2)\bar{t}$$
$$2\sin 2r\bar{t}\cos 2\bar{t} = \sin(2r+2)\bar{t} + \sin(2r-2)\bar{t}$$

$$pA_0 - qA_2 = 0$$
$$(p-4)A_2 - q(2A_0 + A_4) = 0$$
$$\cdots\cdots\cdots$$
$$(p - 4r^2)A_{2r} - q(A_{2r-2} + A_{2r+2}) = 0$$

Now, we're going to examine a method that allows us to find the solutions of this equation. Here we are very advanced in the techniques of solving differential equations, we'll go step by step, I'll just show the principle. I am looking for following the discussion we had earlier in the other module on the eigenfunctions of the Mathieu equation, I'm going to look for solutions of period pi. So, I'm going to write a series of cosine terms of the cosine functions which have a period pi, in fact, I've written two r, r equals zero, one, two, three, et cetera, an infinite series of countable terms, and then I've got coefficients A two r, A index two r, these are the respective weights of all these coefficients. If I want to make this e one bar a solution, I have to put it in the differential equation, I would have terms in cosine two t, which multiply the cosine two rt, these cosine products I can reduce them to cosine sum with trigonometric rules that I remind here, when we do this work, it's a bit laborious, we arrive at the following relations: then we have relations, the first ones are easy, but very quickly we see that it spoils, already at the second one in fact, because we have A two, which is linked to A zero and to A four.

Notes

Summary

6m 07s

$$y'' + (p - 2q\cos 2\bar{t}\,)\,y = 0$$

chercher une solution de période π :

$$\bar{e}_1(\bar{t}) = \sum_{r=0}^{\infty} A_{2r}\cos 2r\bar{t}$$

$$2\cos 2r\bar{t}\cos 2\bar{t} = \cos(2r+2)\bar{t} + \cos(2r-2)\bar{t}$$
$$2\sin 2r\bar{t}\cos 2\bar{t} = \sin(2r+2)\bar{t} + \sin(2r-2)\bar{t}$$

$$pA_0 - qA_2 = 0$$
$$(p-4)A_2 - q(2A_0 + A_4) = 0$$
$$\ldots\ldots\ldots\ldots$$
$$(p - 4r^2)A_{2r} - q(A_{2r-2} + A_{2r+2}) = 0$$

$$-\frac{p}{2} = \cfrac{q^2}{4 - p - \cfrac{q^2}{16 - p - \cfrac{q^2}{36 - p - \cfrac{q^2}{(\ldots)}}}}$$

$$\frac{A_2}{A_0} = \frac{p}{q}$$
$$\frac{A_2}{A_0} = \frac{-q}{4 - p + q\frac{A_4}{A_2}}$$
$$\frac{A_{2r}}{A_{2r-2}} = \frac{-q}{4r^2 - p + q\frac{A_{2r+2}}{A_{2r}}}$$

Similarly, we have A two r which is linked to two r minus two, and two r plus two. We have recurrence relations which are quite complex, but finally, from the first one, we can obtain A two over A zero, from the second one I can also calculate A two over A zero, but there is an A four which intervenes. The A four, I can get it from the generalization of that formula, which I get here by writing A two r on A two r minus two. This gives me this term which will depend on A two r plus two. Now, to make this solution consistent, I have to impose a relation between p and q. Indeed, this p over q here which is worth A two over A zero is also worth that. With the A four over A two that I can calculate thanks to this formula. I thus arrive at an infinite fraction, I stopped here to write the terms, fortunately it is a fraction which converges rather quickly, and one can calculate rather easily some terms of this relation between p and q.

Notes

Summary

7m 48s

Fonctions propres

autres solutions de période π :

$$\bar{e}_2(\bar{t}) = \sum_{r=0}^{\infty} B_{2r+2} \sin(2r+2)\bar{t}$$

solutions de période 2π :

$$\bar{e}_3(\bar{t}) = \sum_{r=0}^{\infty} A_{2r+1} \cos(2r+1)\bar{t}$$

$$\bar{e}_4(\bar{t}) = \sum_{r=0}^{\infty} B_{2r+1} \sin(2r+1)\bar{t}$$

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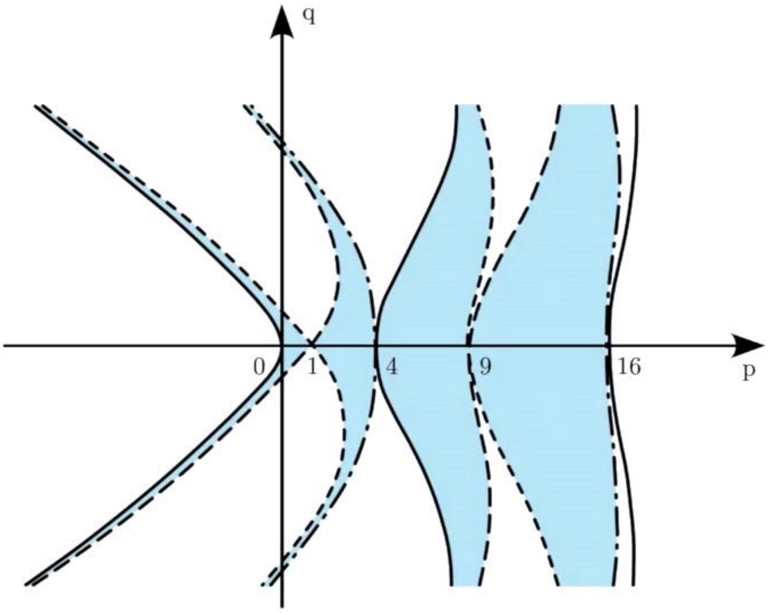
Now, so far, I have taken terms in cosine, indeed I could have taken terms in sine, as indicated here, that will give me a solution in e two bars, and then afterwards I will have to look at the solutions of period, not pi but two pi, that will give me again series cosine and sine series, for each of these functions we have to calculate the relations between p and q to ensure coherence, each time we get another relation between p and q, and now I show you the global result on a p and q diagram, what we have shown is how we find lines such as these.

Notes

Summary

9m 09s

Relation de dispersion p(q) des fonctions propres



$$y'' + (p - 2q\cos 2\bar{t}\,)\, y = 0$$

$$q = \frac{-2d_0}{L}$$

$$p = \frac{g}{L\Omega^2}$$

What I haven't shown is that in the areas marked in blue here, we have stable solutions, and outside we have unstable solutions. So the eigenfunctions of period pi and two pi of Mathieu's equation give us the limits between the unstable and the stable zones. I recall here the notations we used for p and q, p and q which appear on this diagram.

Notes

Summary

9m 56s