http://go.epfl.ch/traite-meca-5-6

Pendules couplés

Mécanique, cours 27.2

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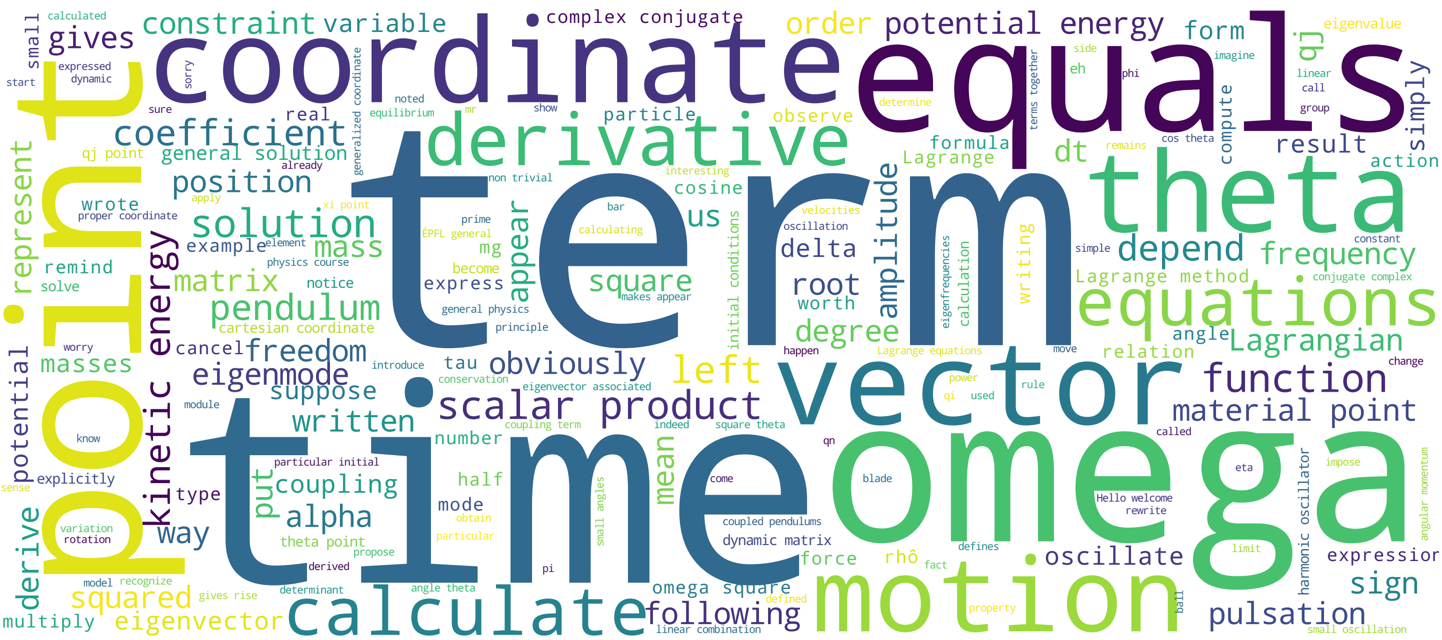

Video

Pendules couplés

- 2 pendules couplés
- Equations du mouvement
- Modes et fréquences propres

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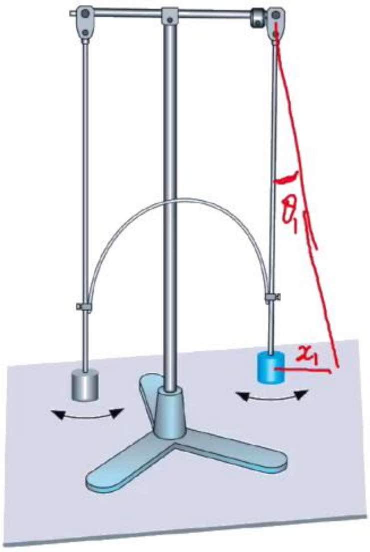
Hello, welcome to the ÉPFL general physics course. In this module, we will look at the particular case of two coupled pendulums, we will look at the small oscillations of these pendulums, we will use Lagrange's method to obtain the equations of motion, and we will look for the eigenmodes and the eigenfrequencies of this system.

Notes

Summary

0m 03s

Pendules couplés : Lagrange



Coordonnées : écarts à l'équilibre x_1 et x_2

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

Pesanteur, énergie potentielle d'un pendule :

$$V_1 = mg\ell(1-\cos\theta_1) \approx mg\ell\left(1 - 1 + \frac{1}{2}\theta_1^2\right)$$

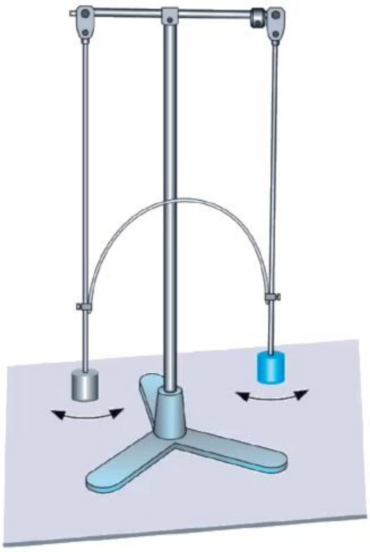
So here is a picture of this system, you have two equal masses, there is a joint here, we are going to neglect the mass of the bar, we are going to suppose that we have material points, and then we have a blade here, a spring blade which defines the coupling between the two pendulums. We'll use Lagrange's method, we suppose that the two pendulums oscillate in a horizontal plane, so we have two degrees of freedom, and we'll take as coordinates the gaps, here we have small angles and each, we can take the cartesian coordinate which represents the oscillation of each of the masses, so we have so a coordinate x one and x two. We will take equal masses, which will greatly simplify the, calculation of the dynamic matrix. The kinetic energy, we can immediately write it, with cartesian coordinates, and now for the potential energy, if we introduce for example, for the pendulum number one here, an angle theta one, we obviously have if we extend here, x one but we're going to take the limit when theta one tends towards zero, eh, there we have a certain angle.

Notes

Summary

0m 24s

Pendules couplés : Lagrange



Coordonnées : écarts à l'équilibre x_1 et x_2

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

Pesanteur, énergie potentielle d'un pendule :

$$V_1 = mg\ell(1-\cos\theta_1) \approx mg\ell\left(1 - 1 + \frac{1}{2}\theta_1^2\right)$$

Petits angles : $\theta_1 \approx \frac{x_1}{\ell}$

Couplage, énergie potentielle (choix) :

$$\frac{1}{2}k(x_1 - x_2)^2$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right)x_1^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right)x_2^2 + kx_1x_2$$

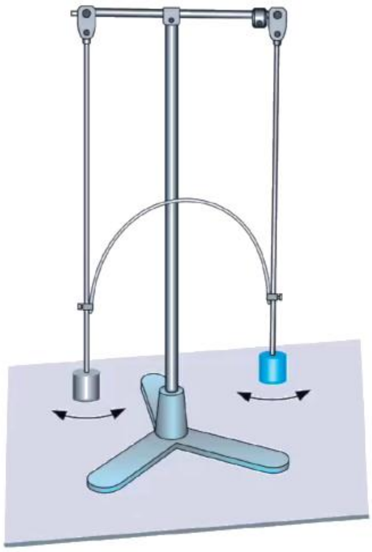
So we're used to writing the potential energy for a pendulum, it's a term like this, now we're going to take small angles, so we're going to make a cosine approximation, and we have to go and find the second order, we have a minus for the cosine and a minus one half of theta one squared, with the minus it becomes a plus, and if we now write that the theta one angle is x one on L, it's a very simple trigonometry relation that we often used, we can write the potential energy and the kinetic energy with the theta one coordinate. For the coupling which is defined by this vibrating plate, we will model the system with the simplest expression we can imagine to represent this action, and I propose to simply take a term like this, we will see that indeed this term gives rise to a coupling which, which has, which is linear, which gives rise to linear equations, and which represents well the effects which we try to model. To calculate the Lagrangian, I'm going to express it with the coordinates x one, x two, so I have the kinetic energy here, I have the potential energy of the mass one, here, expressed with the coordinate x one, the potential for mass two has the same form, it's this term here, and then the coupling term, it appears here with a k term x one, x two, the term in x one squared, I've grouped it with the term in mg, like this, for the particle one and for the particle two.

Notes

Summary

2m 03s

Pendules couplés : Lagrange



Coordonnées : écarts à l'équilibre x_1 et x_2

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$$

Pesanteur, énergie potentielle d'un pendule :

$$V_1 = mg\ell(1 - \cos\theta_1) \approx mg\ell\left(1 - 1 + \frac{1}{2}\theta_1^2\right)$$

Petits angles : $\theta_1 \approx \frac{x_1}{\ell}$

Couplage, énergie potentielle (choix) :

$$\frac{1}{2} k(x_1 - x_2)^2$$

$$L = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right) x_1^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right) x_2^2 + kx_1x_2$$

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Here is my Lagrangian for a system with two degrees of freedom, with coordinates x one and x two.

Notes

Summary

4m 17s

Pendules couplés : équations du mouvement

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right)x_1^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right)x_2^2 + \underline{kx_1x_2}$$

$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \qquad \frac{\partial L}{\partial x_1} = -\left(k + \frac{mg}{\ell}\right)x_1 + kx_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m\ddot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$m\ddot{x}_1 + \left(k + \frac{mg}{\ell}\right)x_1 - kx_2 = 0$$

$$m\ddot{x}_2 + \left(k + \frac{mg}{\ell}\right)x_2 - kx_1 = 0$$

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Now it's a matter of calculating the equations of motion using the Lagrange equations, so, to get there, here's the result. To get there, what do we have to do? We have to calculate d of L, sorry, d of L over d of x a, point, so obviously we have an m x a point here. And then we have to do the derivative with respect to time, of d of L on d of x one point, that will obviously give us m x one point point, it's very simple, we have to compute d of L on d of x one, then there's a term that comes here, k plus mg on L, times x one, with a minus sign, and that term there also gives a k x two term. Now, d of, d on dt of d of L on d of xi point, minus d of L on d of xi equals zero, that's the Lagrange equation, so we have that term there derived by with respect to time, minus that term there which is equal to zero, and the same thing for the x two coordinate, so we have that term there. You notice that if we didn't have this coupling term, if there wasn't this term, here we would have an equation for x one only, and there for x two only, we could solve for one and for the other independently, which complicates things and which makes physics more interesting and which accounts for the phenomena we observe when we study coupled pendulums, it's this term. This term couples the two equations of motion.

Notes

Summary

4m 27s

Fréquences propres

$$m\ddot{x}_1 + \left(k + \frac{mg}{\ell}\right) x_1 - kx_2 = 0$$

$$m\ddot{x}_2 + \left(k + \frac{mg}{\ell}\right) x_2 - kx_1 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

$$m\omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k + \frac{mg}{\ell} & -k \\ -k & k + \frac{mg}{\ell} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{vmatrix} = 0$$

$$0 = \left(k + \frac{mg}{\ell} - m\omega^2\right)^2 - k^2 = \left(\cancel{k} + \frac{mg}{\ell} - m\omega^2 \cancel{-k}\right)\left(k + \frac{mg}{\ell} - m\omega^2 + k\right)$$

$$\omega_1 = \pm\sqrt{\frac{g}{\ell}} \qquad \omega_2 = \pm\sqrt{\frac{g}{\ell} + \frac{2k}{m}}$$

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So, following the general method, we have here the equations of motion, we're going to look for solutions, the eigenmodes which are vectors x one, x two, which oscillate at a given omega pulsation. So we write amplitudes a one and a two, we drift twice with respect to time, which makes appear a minus omega square, so here I put plus because I passed these terms on the other side of the equal sign, I write that in the form, these equations, these terms there in matrix form, and so I have here my dynamic matrix. Now we simply have a coefficient m, we don't need to worry about the existence of this coefficient. I can group all these terms together, if I want solutions to one and two non-trivial, this matrix must have a zero determinant. Which gives the following characteristic equation: it's this term times this term minus k squared, which I can write like this, eh, we have the same terms here and there. This is of the type a square minus b square, I can write it a minus b times a plus b. Here I see the solutions appear. So I have, if this term is zero, I have an omega one which is worth, these two terms canceling out, I am left with simply omega equals more or less root of g on L, and here I have two k.

Notes

Summary

6m 42s

Fréquences propres

$$m\ddot{x}_1 + \left(k + \frac{mg}{\ell}\right) x_1 - kx_2 = 0$$

$$m\ddot{x}_2 + \left(k + \frac{mg}{\ell}\right) x_2 - kx_1 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

$$m\omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k + \frac{mg}{\ell} & -k \\ -k & k + \frac{mg}{\ell} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{vmatrix} = 0$$

$$0 = \left(k + \frac{mg}{\ell} - m\omega^2\right)^2 - k^2 = \left(\cancel{k} + \frac{mg}{\ell} - m\omega^2 \cancel{-k}\right)\left(k + \frac{mg}{\ell} - m\omega^2 + k\right)$$

$$\omega_1 = \pm\sqrt{\frac{g}{\ell}} \qquad \omega_2 = \pm\sqrt{\frac{g}{\ell} + \frac{2k}{m}}$$

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So I have this root there that appears. And again, we have a plus or minus. One might be surprised to have here a term that depends on k, but there a term that does not depend on k. When we have calculated the amplitudes a one and a two, we will understand better why one of the modes does not depend on k.

Notes

Summary

8m 40s

Modes propres

$$\begin{pmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\omega_1 = \pm\sqrt{\frac{g}{\ell}}$$

$$ka_1 - ka_2 = 0$$

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2 = \pm\sqrt{\frac{g}{\ell} + \frac{2k}{m}}$$

$$-ka_1 - ka_2 = 0$$

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

I rewrite my characteristic equation, I have two solutions, if now I take omega one in this equation, from the first equation here, I have this term which cancels with this one, I am left with k times a one, minus k times a two, equals zero. That means a one equals a two. So my eigenvector associated with this eigenvalue is one, one, which we could normalize if we wanted, for this mode if we put omega square in there, I have k minus two k. That's k minus k. So now we have minus k times a one, minus k times a two, which is zero. What I wrote here. Here we have a one which is minus a two. a one minus a two. So we have eigenvectors like this. This is the eigenvector associated with this eigenvalue.

Notes

Summary

9m 06s

Solution générale et coordonnées propres

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1\, e^{+i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_{-1}\, e^{-i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2\, e^{+i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_{-2}\, e^{-i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A_1 = A^*_{-1} \qquad A_2 = A^*_{-2}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \cos(\omega_1 t + \phi_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cos(\omega_2 t + \phi_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$x_1 + x_2$: pulsation ω_1

The general solution can be written like this: the general solution for x one, x two, is a linear combination of the eigenmodes with frequency plus omega one or minus omega one and eigenvector one, one. We also have plus omega two, minus omega two with eigenvector one, minus one. We want real solutions. So this whole expression has to be x one and x two real. One way to do it is to make sure to take for a minus one the complex conjugate of a one, because there we already have the complex conjugate, the, these terms are complex conjugates of each other. If we also make sure that a minus one is the conjugate complex of a one, then we have terms that are of the order of the type, a complex number plus its conjugate complex, that makes twice the real part of this number, and that is real. So, we will impose this rule on the coefficients a one and a minus one and a two and a minus two, to ensure reality. We could also have written x one, x two explicitly with real functions by introducing phases as we did for the simple harmonic oscillator. We notice that if I compute x one plus x two, these terms cancel each other out. So, we are left with terms that oscillate at the omega one frequency.

Notes

Summary

10m 10s

Solution générale et coordonnées propres

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1\, e^{+i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_{-1}\, e^{-i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2\, e^{+i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_{-2}\, e^{-i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A_1 = A^*_{-1} \qquad A_2 = A^*_{-2}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \cos(\omega_1 t + \phi_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cos(\omega_2 t + \phi_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$x_1 + x_2$: pulsation ω_1

$x_1 - x_2$: pulsation ω_2

So, we see this idea appearing that we have a proper coordinate which has a well defined omega one pulsation. In the same way, if we make x one minus x two, it's these terms that cancel each other out, there remains these two, this contribution, which is at the omega two frequency, the omega two pulsation, the proper coordinate x one minus x two has the omega two pulsation.

Notes

Summary

11m 55s

Conditions initiales, projections

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1\, e^{+i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_{-1}\, e^{-i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2\, e^{+i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_{-2}\, e^{-i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A_{-1} = A_1^*$$
$$A_{-2} = A_2^*$$

Position : $$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = (A_1 + A_{-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (A_2 + A_{-2}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(1\ 1)\cdot \implies x_{10} + x_{20} = 2(A_1 + A_{-1})$$
$$(1\ -1)\cdot \implies x_{10} - x_{20} = 2(A_2 + A_{-2})$$

Vitesse : $$\begin{pmatrix} v_{10} \\ v_{20} \end{pmatrix} = +i\omega_1\,(A_1 - A_{-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\omega_2\,(A_2 - A_{-2}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If now, we have a problem that is given to us with particular initial conditions, there I remind you the general solution with these conditions are the coefficients. If now we are given the position at t equals zero with x one, zero, x two, zero, if I take t equals zero in there, I have ones, I have these terms left. And to find the coefficients, I can simply multiply that equation, multiply in the sense of the scalar product with one times the vector one, one and once with the vector one, minus one, which I have noted here. One, one, scalar product with the equation. That gives you, x one of zero plus x two of zero, which is a one plus a minus one times two, that's the scalar product of one, one with one, one, that's two. If I do the scalar product with one, minus one, I put one minus one here, scalar product. I do the same scalar product everywhere, maybe we can do it like that, I make the scalar product act on it, on it, on it, I have this relation there. If I now have an initial condition on the velocities, then we have to derive this expression with respect to time, that makes appear an i, omega one here, minus i, omega one there, i, omega two, minus i, omega two.

Notes

Summary

12m 18s

Conditions initiales, projections

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1\, e^{+i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_{-1}\, e^{-i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2\, e^{+i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_{-2}\, e^{-i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{array}{l} A_{-1} = A_1^* \\ A_{-2} = A_2^* \end{array}$$

Position : $$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = (A_1 + A_{-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (A_2 + A_{-2}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{aligned} (1\ 1)\cdot \implies & \quad x_{10} + x_{20} = 2(A_1 + A_{-1}) \\ (1\ -1)\cdot \implies & \quad x_{10} - x_{20} = 2(A_2 + A_{-2}) \end{aligned}$$

Vitesse : $$\begin{pmatrix} v_{10} \\ v_{20} \end{pmatrix} = +i\omega_1\, (A_1 - A_{-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\omega_2\, (A_2 - A_{-2}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{aligned} (1\ 1)\cdot \implies & \quad v_{10} + v_{20} = 2i\omega_1 \big(A_1 - A_{-1}\big) \\ (1\ -1)\cdot \implies & \quad v_{10} - v_{20} = 2i\omega_2 \big(A_2 - A_{-2}\big) \end{aligned}$$

You then take t equals zero, the exponentials disappear. You're left with these terms. And to solve, you multiply by the vector one, one or the vector one, minus one. Multiplying by the vector one, one, we get v one, zero plus v two, zero. With one, minus one, we have v one of zero minus v two of zero. And there we have our four equations for our four unknowns and we can determine the motion for these particular initial conditions.

Notes

Summary

13m 54s