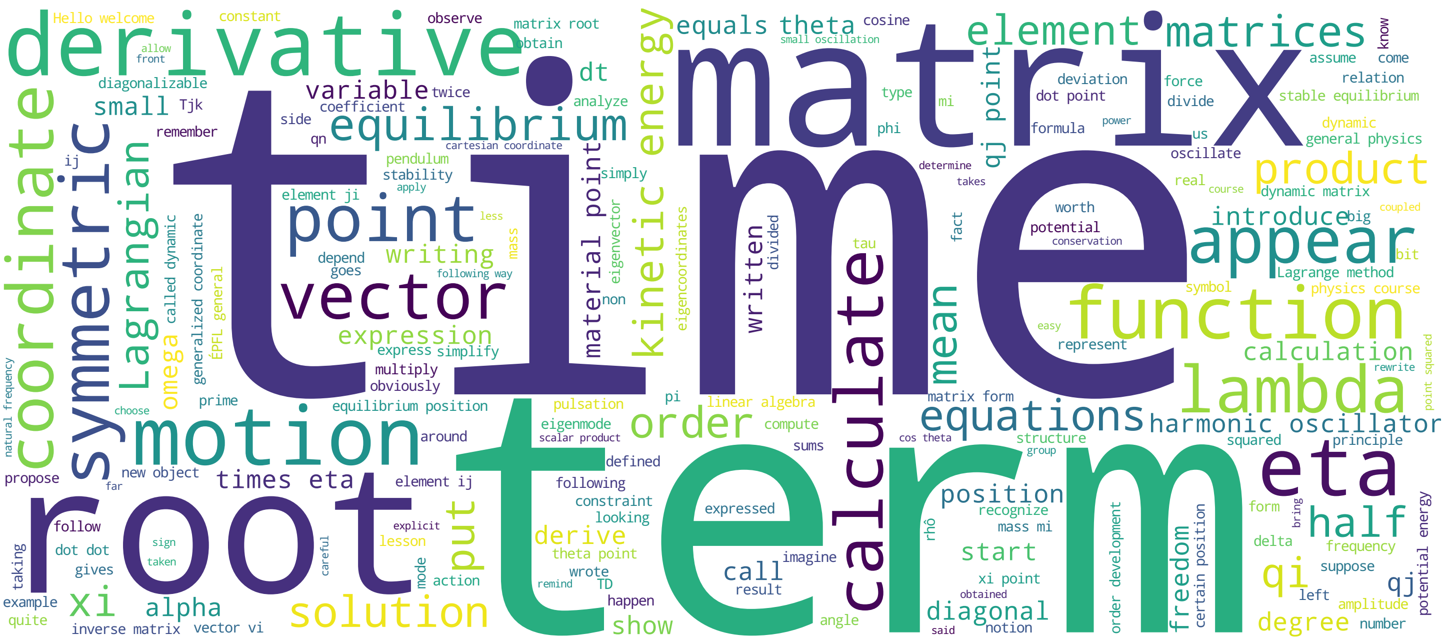
Systèmes vibratoires discrets, linéaires

Mécanique, cours 27.1

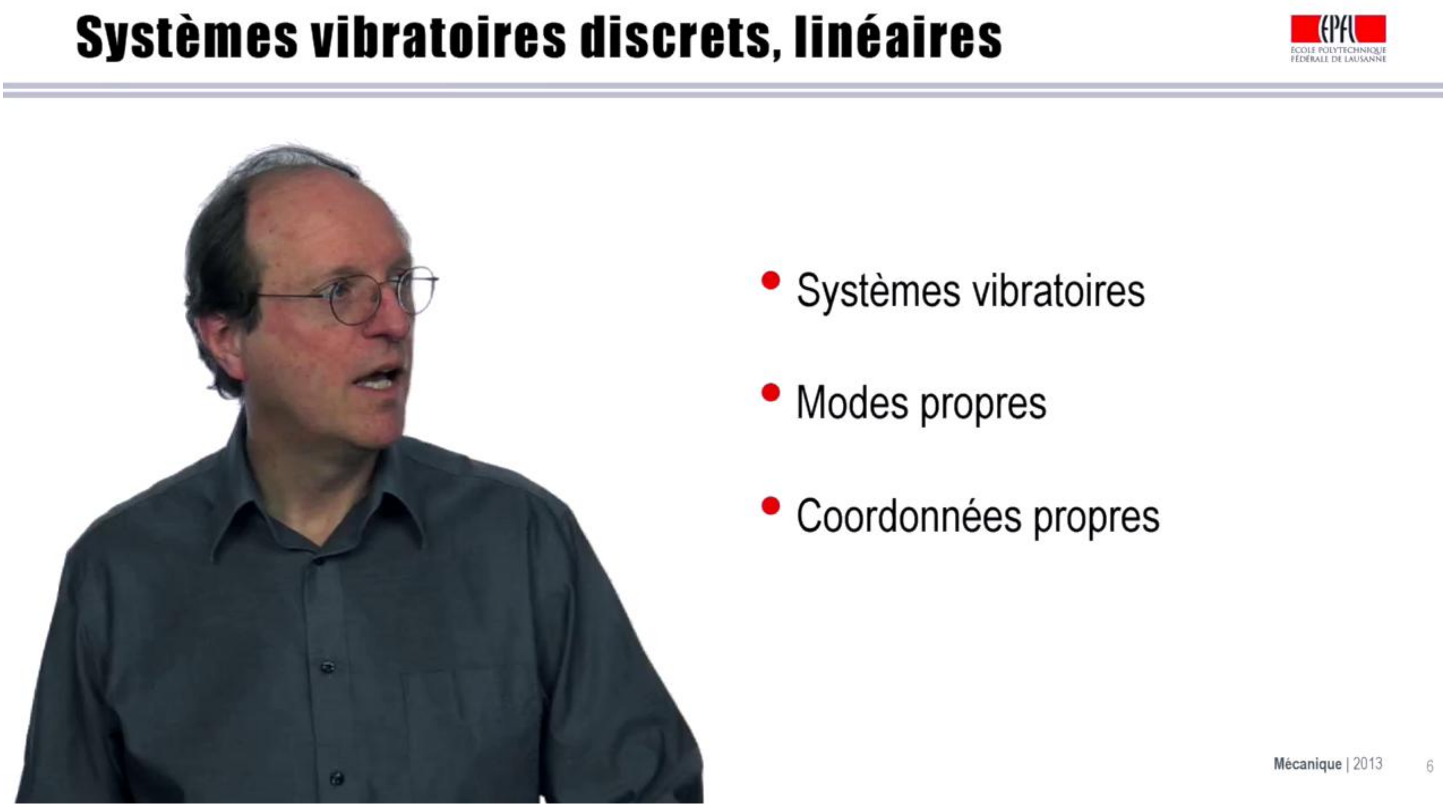
Jean-Philippe Ansermet

http://go.epfl.ch/traite-meca-5-29



Video

EPFL



Hello, welcome to the ÉPFL general physics course. In this lesson, I would like to analyze the dynamics of a harmonic oscillator system, coupled. I will start by defining my vibratory system, the coordinates I will use, I will use Lagrange's method to obtain the equations of motion, I will then show how these equations of motion can be expressed in a matrix fashion, and I will define the eigenmodes and eigenfrequencies characteristic of the system. I will finish with the definition of what we call the eigencoordinates of our dynamical system.

Notes

Summary

0m 03s

Définition : systèmes vibratoires discrets, linéaires

N points matériels, masses m_i $(i = 1 \dots N)$

n degrés de liberté

$$x_i = x_i(q_1, \dots, q_n),\ i = 1, \dots, 3N$$

Le système est en équilibre stable à $(q_{10}, q_{20}, \dots, q_{n0})$

Petites oscillations autour de cet équilibre

$$\eta_i = q_i - q_{i0} \to 0$$

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I imagine that I have N material points, big N material points of mass mi. That I have small n degrees of freedom and I choose the coordinates q one and qn, generalized coordinates to define the state of my system. So I have the cartesian coordinates xi, of the mass mi which are given in function of q un and qn. I suppose, be careful, this is a hypothesis that we make, about our dynamic system, we suppose that this system has a stable equilibrium, has the position q one zero, q two zero, qn zero. And we'll look at the small oscillations of the system around this stable equilibrium. So if we define the deviation of the coordinate from the equilibrium position, we assume that we have taken the limit when this deviation tends to zero.

Notes

Summary

0m 48s

Application de la méthode de Lagrange

Energie cinétique

$$x_i = x_i(q_1, \ldots, q_n),\ i = 1, \ldots, 3N \qquad \dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j$$

$$T = \frac{1}{2} \sum_{i=1}^{3N} m_i \dot{x}_i^2 \quad = \frac{1}{2} \sum_{i=1}^{3N} m_i \left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j \right) \left(\sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} \dot{q}_k \right)$$

$$T_{jk} = \sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k} \qquad T = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} \dot{q}_j \dot{q}_k$$

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Then I propose to use Lagrange's method to obtain the equations of motion, and I start with the kinetic energy. Then, since xi is a function of qi, and qi, each qi is a function of time, if I want to compute xi point, I have to compute the derivative of xi with respect to the qi, and the derivative of the qi with respect to time. Formally, it goes like this. I need to calculate the derivative of xi with respect to qj, times d of qj over dt, and I sum over all the coordinates here. This gives me xi point. The kinetic energy is half the mass times the velocity squared, so there is the xi point squared, I use this formula, I rewrite it like this taking the precaution to write once sum over j, and sum over k, to distinguish well these two sums. Now in this expression of the kinetic energy, I have a sum on i here, of the mi, the xi, which appear here too. I can group all these terms together, and if I call Tjk, the sum over i up to three N, of the mi times these terms, times this, then T can be written a half, that's the half I have here, Tjk, qj point, that's this qj point, qk point, that's this one. Now let's go to the potential energy.

Notes

Summary

1m 53s

Energie potentielle

$$V(q_1, q_2, \ldots, q_n) = \; V(q_{10}, \ldots, q_{n0}) \; + \sum_i \left.\frac{\partial V}{\partial q_i}\right|_{\text{éq}} \eta_i \quad + \frac{1}{2} \sum_{\ell m} \left.\frac{\partial^2 V}{\partial q_\ell \partial q_m}\right|_{\text{éq}} \eta_\ell \, \eta_m$$

$$\eta_i = q_i - q_{i0}$$

So, here, we're going to bring in physics when it's given about this system, we said we had a stable equilibrium, at a certain position. Now we have to express the existence of this stability, at a certain position. To do this, I will make a second order approximation for the potential V, around the equilibrium position. At zero order, I will say that V is a function, and simply equal to the value at equilibrium. With that, we won't go very far, so we'll push the development further. First we'll do a first order development. The first order development, for the variables eta i which are the deviation from the equilibrium position, would be terms like this. Now, we say that the value at equilibrium, at equilibrium, well, at this position qi zero, we are at equilibrium. So these terms are zero. That's what determines the equilibrium. So we have to go to the second order. Here is the second order which appears as this, with second derivatives with respect to all the variables, times the difference eta l, eta m, corresponding to the variables ql and qm.

Notes

Summary

3m 41s

Application de la méthode de Lagrange

Energie potentielle

$$V(q_1, q_2, \ldots, q_n) = V(q_{10}, \ldots, q_{n0}) + \sum_i \left.\frac{\partial V}{\partial q_i}\right|_{\text{éq}} \eta_i + \frac{1}{2} \sum_{\ell m} \left.\frac{\partial^2 V}{\partial q_\ell \partial q_m}\right|_{\text{éq}} \eta_\ell \, \eta_m$$

$$\eta_i = q_i - q_{i0} \qquad V_{\ell m} = \left.\frac{\partial^2 V}{\partial q_\ell \partial q_m}\right|_{\text{éq}} \qquad V(q_1, q_2, \ldots, q_n) = \frac{1}{2} \sum_{\ell m} V_{lm} \, \eta_\ell \, \eta_m$$

$$T = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{jk} \dot{\eta}_j \dot{\eta}_k \qquad T_{jk} = \sum_{i=1}^{3N} m_i \left.\frac{\partial x_i}{\partial q_j}\right|_{\text{éq}} \left.\frac{\partial x_i}{\partial q_k}\right|_{\text{éq}}$$

Lagrangien : $$L = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(T_{jk} \dot{\eta}_j \dot{\eta}_k - V_{jk} \eta_j \eta_k \right)$$

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If now I write this coefficient Vlm, and I remember that this term is null, and this one is a constant term which brings nothing, in the Lagrangian because what counts with the Lagrangian is the derivatives that we are going to do with it, so we can drop this term, and I can write that the V of q one, qn, is the one half, which was here, sum on l, and on m, going up to n, I could have written here n, the number of degrees of freedom, of the Vlm, times the eta l, times the eta m. For the kinetic energy, we had obtained this expression, now these eta are small, if we want an expression of T at the second order, we have to take for Tjk these derivatives, also evaluated at equilibrium. These terms are of the second order, these terms must be of the zero order. So I take them at equilibrium, Tjk are now constants. And my Lagrangian L is T minus V, it's this term minus this one. What I wrote here.

Notes

Summary

5m 16s

Vecteur : $\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_\ell \end{pmatrix}$

Matrice T : $T_{jk} = \sum_{i=1}^{3N} m_i \left.\frac{\partial x_i}{\partial q_j}\right|_{\text{éq}} \left.\frac{\partial x_i}{\partial q_k}\right|_{\text{éq}}$

Matrice V : $V_{\ell m} = \left.\frac{\partial^2 V}{\partial q_\ell \partial q_m}\right|_{\text{éq}}$

$$T\ddot{\boldsymbol{\eta}} = -V\boldsymbol{\eta}$$

To apply Lagrange's method, we start from the Lagrangian, and we have to derive L with respect to eta i, and then this derivative with respect to time, so when we derive with respect to eta i, we have twice a term that appears, of type Tik, and the derivative of eta k, is simply eta k point, it's eta k point. So there is no big problem to calculate the first derivative and then derive it by time, we just have terms like that, we have a sum, and here we have the same argument, we have a double sum, we can take a term, as we have a term that appears for each of the sums, we have twice the same term, so the term two, the term one half, falls, and we have this, which is indexed by i, and we have i which is one to n, so we have n equations of motion. Now, this way of writing, is cumbersome and not very explicit, so we're going to do a writing transformation that allows us to, write the equation of motion in matrix form. First, the gaps eta one to eta l, I constitute them in vectors. The Tjk's, I'll consider them as elements of the matrix, of a matrix T. And the Vlm, as the elements of a matrix V. So, my equation of motion, I can write it T times eta point point, this vector derived twice with respect to time, equals less V times eta.

Notes

Summary

6m 46s

Application de la méthode de Lagrange

$$\text{Vecteur :} \quad \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_\ell \end{pmatrix}$$

$$\text{Matrice T :} \quad T_{jk} = \sum_{i=1}^{3N} m_i \left.\frac{\partial x_i}{\partial q_j}\right|_{\text{éq}} \left.\frac{\partial x_i}{\partial q_k}\right|_{\text{éq}}$$

$$\text{Matrice V :} \quad V_{\ell m} = \left.\frac{\partial^2 V}{\partial q_\ell \partial q_m}\right|_{\text{éq}}$$

$$T\ddot{\boldsymbol{\eta}} = -V\boldsymbol{\eta}$$

We recognize more or less the structure of the equation of a harmonic oscillator, for the eta vector, it's not quite that because of the T. You could pass the T on the other side, of the equals sign, but in what follows, I'm going to show you a more clever technique, which ensures that the dynamic matrix is symmetric, which will simplify the calculations that follow.

Notes

Summary

8m 44s

Diagonalisation

$$T\ddot{\boldsymbol{\eta}} = -V\boldsymbol{\eta}$$

$$T_D = \begin{pmatrix} T_{11} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & T_{nn} \end{pmatrix}$$

T réelle et symétrique $\Longrightarrow$ T est diagonalisable : $T = U^{-1} T_D U$

U orthogonale ($U^{-1}_{ij} = U_{ji}$)

$$T = U^{-1}\sqrt{T_D}\,\sqrt{T_D}\,U = \underbrace{U^{-1}\sqrt{T_D}U}_{\sqrt{M}}\underbrace{U^{-1}\sqrt{T_D}\,U}_{\sqrt{M}} = \sqrt{M}\,\sqrt{M}$$

So to do this, to express this equation of motion in a way closer than that of a harmonic oscillator, as T is symmetric, I know that T can be diagonalized, this is a known presumed result learned in a linear algebra course, if T is diagonalizable, it means that there exists a matrix U such that T can be written as U minus one times T diagonal times U, TD means T diagonal, I mean that TD is a matrix with non-zero elements on the diagonal, and zeros everywhere else, and U, it's a result also of linear algebra, is an orthogonal matrix, it means that the inverse is computed by computing a transpose. Here you have U minus one, ij, that's just Uji. Now I'm going to write, it's a bit of a funny way of writing, we're going to introduce the notion of root of the matrix, obviously it's about taking the square root of the elements on the diagonal, here in between I'm going to introduce U, U minus one, U times U minus one that's one, here's U, U minus one, and there I see U minus one appear, root of TD U, I can write it if I call that matrix root of M, again, it is a symbol, it's a writing, I have root of M times root of M, I have twice the same thing. So I introduce a matrix root of M by this calculation.

Notes

Summary

9m 17s

Diagonalisation

$$T\ddot{\boldsymbol{\eta}} = -V\boldsymbol{\eta}$$

$$T_D = \begin{pmatrix} T_{11} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & T_{nn} \end{pmatrix}$$

T réelle et symétrique $\Longrightarrow$ T est diagonalisable : $T = U^{-1} T_D U$

U orthogonale ($U^{-1}_{ij} = U_{ji}$)

$$T = U^{-1}\sqrt{T_D}\,\sqrt{T_D}\,U \;\; = U^{-1}\sqrt{T_D}UU^{-1}\sqrt{T_D}\,U \; = \sqrt{M}\,\sqrt{M}$$

$$\sqrt{M} = U^{-1}\sqrt{T_D}\,U$$

$\sqrt{M}$ est symétrique $\quad (\sqrt{M})_{ij} = \sum_{lm} U^{-1}_{il}(\sqrt{T_D})_{lm}U_{mj} \; = \sum_{lm} U^{-1}_{jm}(\sqrt{T_D})_{ml}U_{li} = (\sqrt{M})_{ji}$

$$\frac{1}{\sqrt{M}} = \sqrt{M}^{-1} \text{ est aussi symétrique}$$

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With this equation that defines root of M. Now I show that root of M is symmetric. Then I calculate the element ij of the matrix root of M, root of M is defined by this product of three matrices, this product of three matrices, if I write the components of the matrices, I have a he, a lm, a mj, and I sum over l, and I sum over m. Now, Umj is equal to Ujm minus one. Because of this orthogonality property. That's what I wrote here. Uil minus one is equal to Uli, and here I observe that I have the product of the matrix U minus one, times the root matrix of TD, times the matrix U, and I have the element ji. So this is root of M index ji. So here I have ij and ji, so this matrix of M is symmetric. I will define an even more curious symbol for matrices, but this symbol only represents the inverse matrix of the root of M, and it is easy to check that the inverse matrix is also symmetric.

Notes

Summary

11m 19s

$$T\ddot{\boldsymbol{\eta}} = -V\boldsymbol{\eta}$$

$$\sqrt{M}\,\ddot{\boldsymbol{\eta}} = -\frac{1}{\sqrt{M}}V\frac{1}{\sqrt{M}}\sqrt{M}\,\boldsymbol{\eta}$$

$$\boldsymbol{X} = \sqrt{M}\boldsymbol{\eta} \qquad D = \frac{1}{\sqrt{M}}V\frac{1}{\sqrt{M}}$$

$$\ddot{\boldsymbol{X}} = -D\boldsymbol{X}$$

D est symétrique : $$D_{ij} = \sum_{lm}\left(\sqrt{M}^{-1}\right)_{il}V_{lm}\left(\sqrt{M}^{-1}\right)_{mj} = D_{ji}$$

All these calculations under these devices which define these root matrices of M, or a super-root of M, will simplify my equations in the following way: Instead of writing my equation of motion like this, I'm going to here put a root, T I'm going to write it as root of M times root of M, and I'm going to divide by, well divide, it's multiplied by the inverse matrix, of root of M, so I have a on root of M that comes here, here I have this product that's worth one, that I've introduced between V and eta, it's a little game on matrices. Which has the advantage of making this new object appear, let's say this new object, and its second derivative with respect to time, this object I will call big X, it's a vector in the same vector space, it's root of M times eta, and now if I define the so-called dynamic matrix D as being that object, my equation of motion now takes that form, which resembles the equation of the harmonic oscillator, but it's an equation for a vector and D is a matrix. This matrix is symmetric, in fact if I write the definition of the element, I use the definition of D to calculate the element ij of this matrix, I have a product of three matrices, I have a sum on l, I have a sum on m, it appears here, and m appears there, as this matrix is symmetric, I can write jm, and put it there in front, to help the reading.

Notes

Summary

12m 52s

Diagonalisation

$$T\ddot{\boldsymbol{\eta}} = -V\boldsymbol{\eta}$$

$$\sqrt{M}\,\ddot{\boldsymbol{\eta}} = -\frac{1}{\sqrt{M}}\,V\,\frac{1}{\sqrt{M}}\sqrt{M}\,\boldsymbol{\eta}$$

$$\boldsymbol{X} = \sqrt{M}\boldsymbol{\eta} \qquad D = \frac{1}{\sqrt{M}}\,V\,\frac{1}{\sqrt{M}}$$

$$\ddot{\boldsymbol{X}} = -D\boldsymbol{X}$$

D est symétrique : $$D_{ij} = \sum_{lm}\left(\sqrt{M}^{-1}\right)_{il} V_{lm} \left(\sqrt{M}^{-1}\right)_{mj} = D_{ji}$$

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This matrix here is also symmetrical, so I can write li and put it there. And so we have the element ji of the matrix D. Dij equals Dji. The matrix is symmetric. So I have an equation of motion in matrix form with a matrix called dynamic matrix, which is symmetric and real of course.

Notes

Summary

14m 54s

Définition : modes propres, fréquences propres

$$\ddot{\boldsymbol{X}} = -D\boldsymbol{X}$$

$$\ddot{\boldsymbol{v}}_i = -\lambda_i \boldsymbol{v}_i$$

$$\frac{\sqrt{\lambda_i}}{2\pi} \; : \text{ fréquence propre}$$

$$i = 1 \ldots n$$

$$\boldsymbol{v}_i \; : \text{ mode propre}$$

So now I will define the eigenmodes of my problem, and the eigenfrequencies. This is the dynamic equation, I'm looking for solutions of this type. The vector vi is an eigenvector of my problem in the sense that I'm going to look for a solution of harmonic type with here a lambda number i. And the lambda i, obviously, or if you want the root of lambda is divided by two pi, that is the natural frequency where the root of lambda i is the natural pulsation of this mode. This mode also has a frequency. We have n such modes, which can be degenerated, some of them can have the same value, but there are n, and vi, the vector vi, is what we call the eigenmode of this dynamic problem.

Notes

Summary

15m 19s

Définition : coordonnées propres

$$\ddot{\boldsymbol{X}} = -D\boldsymbol{X}$$

D réelle symétrique $\Longrightarrow$ il existe une matrice O : $D = O^{-1}D_\lambda O$ $\quad D_\lambda$ diagonal

$$O\ddot{\boldsymbol{X}} = -D_\lambda O\boldsymbol{X}$$

Changement de variable : $\boldsymbol{x} = O\boldsymbol{X}$ $\quad \ddot{\boldsymbol{x}} = -D_\lambda \boldsymbol{x}$

$$\ddot{x}_i = -\lambda_i x_i \qquad \lambda_i \ (i = 1 \ldots n)$$

x_i coordonnée propre, oscille avec la pulsation : $\sqrt{\lambda_i}$

$$x_i = (O\boldsymbol{X})_i = \sum_k O_{ik}X_k = \sum_k O_{ik}(\sqrt{M}\boldsymbol{\eta})_k = \sum_{k\ell} O_{ik}\sqrt{M}_{k\ell}\,\eta_\ell$$

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now, some authors use the notion of eigencoordinates, this is how we can introduce it: here I have the equation of the dynamics, as D is real and symmetric, it is diagonalizable, it means that there is a matrix O which allows me to write D equals O minus one, D lambda O, D lambda diagonal, and O is orthogonal, I can write, I can put this expression of D here, do, multiply the equation on the left, by O, then I get OX dot dot, here I have this O which comes with this X. I now define the small vector x, as O times the large vector X, I have an equation of type x dot point equals minus D lambda times x, with D lambda a diagonal matrix, and for the coordinate i, you have xi dot point, so a scalar, minus lambda i, xi, you have a harmonic oscillator type equation, for this coordinate small x i, and the pulsation is lambda i, that means that this coordinate oscillates at one frequency. Or at the root pulsation of lambda i. That's what we call the system's own coordinates. If we take up all our definitions, the element i of the vector x, is OX index i, OX is the product of matrix times of a vector, the X, we remember that it was root of M times eta, so here is in term of the eta the ratio, well the relation between the eta and the xi.

Notes

Summary

16m 23s

Définition : coordonnées propres

$$\ddot{\boldsymbol{X}} = -D\boldsymbol{X}$$

D réelle symétrique $\Longrightarrow$ il existe une matrice O : $D = O^{-1}D_\lambda O$ D_λ diagonal

$$O\ddot{\boldsymbol{X}} = -D_\lambda O\boldsymbol{X}$$

Changement de variable : $\boldsymbol{x} = O\boldsymbol{X}$ $\ddot{\boldsymbol{x}} = -D_\lambda \boldsymbol{x}$

$$\ddot{x}_i = -\lambda_i x_i \qquad \lambda_i \ (i = 1 \ldots n)$$

x_i coordonnée propre, oscille avec la pulsation : $\sqrt{\lambda_i}$

$$x_i = (O\boldsymbol{X})_i = \sum_k O_{ik} X_k = \sum_k O_{ik} (\sqrt{M}\boldsymbol{\eta})_k = \sum_{k\ell} O_{ik} \sqrt{M}_{k\ell}\, \eta_\ell$$

The eta were the deviations from the equilibrium for the generalized coordinates that we gave ourselves, and here we have the proper coordinates which oscillate at a given frequency.

Notes

Summary

18m 29s