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Principe de moindre action

Mécanique, cours 26.2

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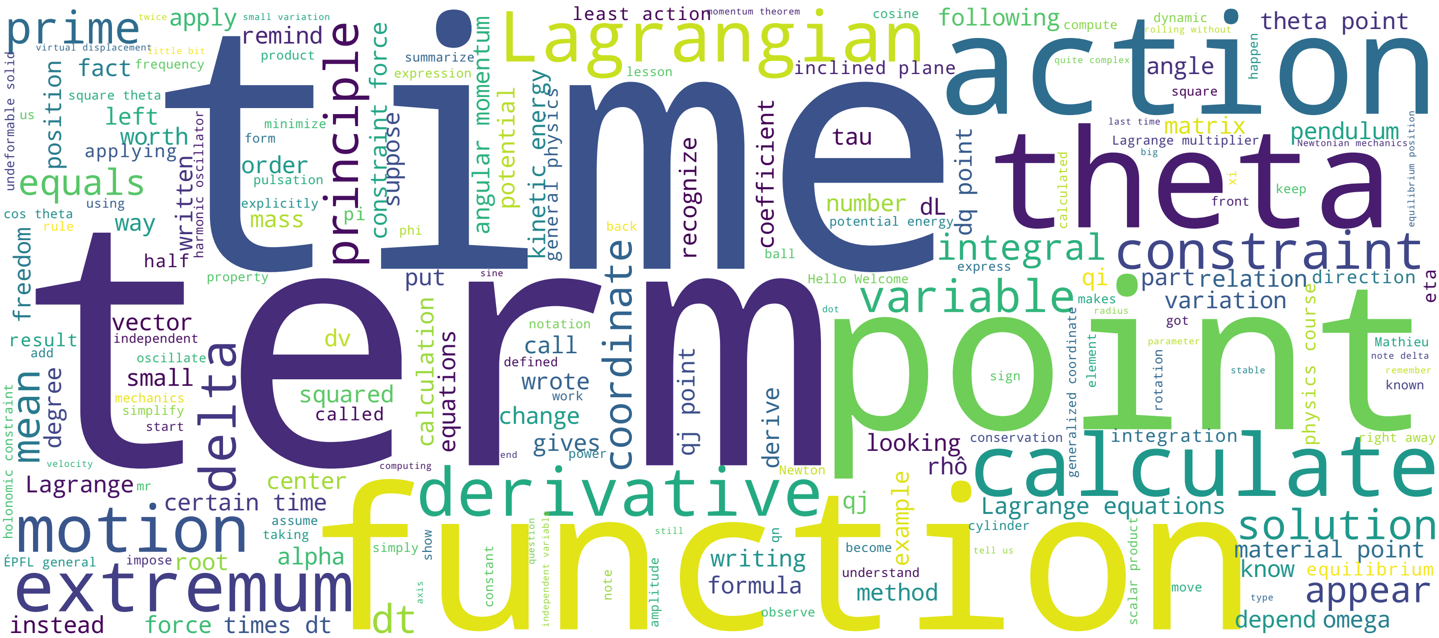

Video

EPFL

Principe de moindre action

Hello. Welcome to the ÉPFL general physics course. In this lesson, I'd like to summarize all the mechanics with a principle that is very quickly stated, it's the principle of, called least action. So first, I'll define action. Then, we're going to see this principle of least action by using the Lagrange equations, and we're going to apply this principle to take the problem, a problem where we had eliminated the constraints, and we're now going to find the constraints by applying this principle. Action.

Notes

Summary

0m 03s

Définition : l'action

$$S = \int_{t_1}^{t_2} L\big(q_1(t), \ldots, q_n(t), \dot{q}_1(t), \ldots, \dot{q}_n(t), t\big)\, dt$$

The action is defined as this. In fact the integral of the Lagrangian is the integral in time taken between a certain time t one, and a certain time t two. Your lagrangian is a function of the coordinates q one to q n which depend on time, from q one point to q n point, and we do not exclude the possibility that the lagrangian also depends explicitly on time. We note S, the action, and that's it. This is our definition of the actian, the action. This is the integral of L when the system evolves from t one to t two.

Notes

Summary

0m 48s

Principe de moindre action

Equations de Lagrange $\Leftrightarrow$ Extremum de l'action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

Soit $q(t)$ la fonction pour laquelle on a l'extremum.

On examine une petite variation $q(t) + \delta q(t)$

On impose $\delta q(t_1) = 0$ et $\delta q(t_2) = 0$.

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So now, we have this extraordinary result which is that the principle, the, the Lagrange equations imply an extremum of the action and conversely. If we have an extremum of the action, we must have the Lagrange equations. We will consider, to simplify the writings, that our Lagrangian depends only on one coordinate. And so, the problem we have to face is to understand what it means to have an extremum of the action, because L is a quite complex object. So to say that L is an extremum is to say the following thing: we suppose that we know q of t, the function for which S is an extremum. Now, if I make a small variation of q, which I note delta q, which is a function of time. If S is extremum, that means if, for q of t. Then the first order, in delta q of t must be zero. Now, we're not going to do this variation any way, we're going to assume the two points, the starting point and the ending point, the points at times t one and t two to be known. We must have a certain initial position, then a position after a certain time. We'll calculate the extremum for two known points.

Notes

Summary

1m 36s

Principe de moindre action

$$\delta S = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)\, dt - L(q,\dot{q},t)\, dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\,\delta q + \underbrace{\frac{\partial L}{\partial \dot{q}}}_{u}\, \underbrace{\delta\dot{q}\Big)\, dt}_{dv}$$

$$\delta S = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\,\delta q\, dt + \left[\frac{\partial L}{\partial \dot{q}}\,\delta q\right]_{t_1}^{t_2} - \int \delta q \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dt$$

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q\, dt$$

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So let's do the calculation. I want to compute a variation of the action, when instead of computing the action for q which is supposed to be the extremum, I take q plus a variation delta q with the restrictions at t one and t two. To do this calculation, I first recognize here that I have a derivative of L with respect to q, point delta q, and I have a derivative of L with respect to q point, times an increase delta q point. Now I'm going to work this second term of the integral, I'm going to do an integration by parts. I have written down these functions u and dv, and the integral of u dv is uv minus the integral of vd u. That's the formula I'm going to apply here: integration by parts. So, if this is the d-function of v, the v-function is delta q, and the u-function is this one. I don't change this term. And you have to make less integral vd u, so it's delta q, Ld u, that makes d on dt of d from dL on dq point times dt. To calculate d of u, I do d over dt of that times dt. Now, we've given ourselves a rule of thumb that we're going to do delta q variations with delta q being zero at t one and t two. So, this term falls. I am left with these two d terms of L on dL q, L minus d on dt of dL on dq point.

Notes

Summary

3m 16s

Principe de moindre action

$$\delta S = \int_{t_1}^{t_2} L\big(q+\delta q, \dot q + \delta \dot q, t\big)\,dt - L\big(q,\dot q,t\big)\,dt \;=\; \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\,\delta q + \underbrace{\frac{\partial L}{\partial \dot q}}_{u}\,\underbrace{\delta \dot q \Big)\,dt}_{dv}$$

$$\delta S = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\,\delta q\,dt + \left[\frac{\partial L}{\partial \dot q}\,\delta q\right]_{t_1}^{t_2} - \int \delta q \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) dt$$

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right] \delta q\,dt$$

Pour tout $\delta q(t) \implies$ Lagrange

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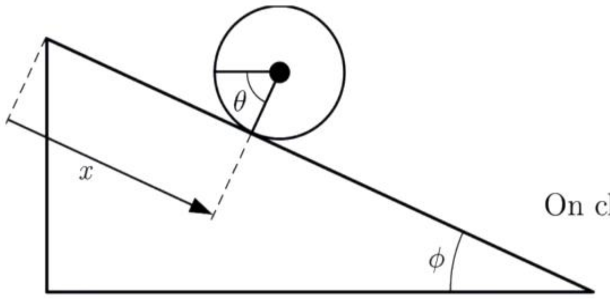
Now, S is extremum for the function q, the solution we are looking for means that we must be able to take any infinitesimal delta q, and we must get zero to first order. The only way to do this is that this term must be zero, so we have Lagrange. And we can do the reasoning backwards, in reverse and find that Lagrange implies the extremum of the action. So there you have it, we got that the motion is the one that minimizes the action.

Notes

Summary

5m 28s

Application : forces de contraintes



$$L = \frac{1}{2} I_\Delta \dot{\theta}^2 + \frac{1}{2} M\dot{x}^2 + Mgx\sin\phi$$

On garde deux degrés de liberté.

On cherche un extremum de l'action sous la contrainte $x = R\theta$.

Méthode des multiplicateurs de Lagrange : on considère

$$L' = \frac{1}{2} I_\Delta \dot{\theta}^2 + \frac{1}{2} M\dot{x}^2 + Mgx\sin\phi - F(x - R\theta)$$

$$\frac{\partial L'}{\partial \dot{\theta}} = I_\Delta \dot{\theta} \qquad \frac{\partial L'}{\partial \theta} = FR$$

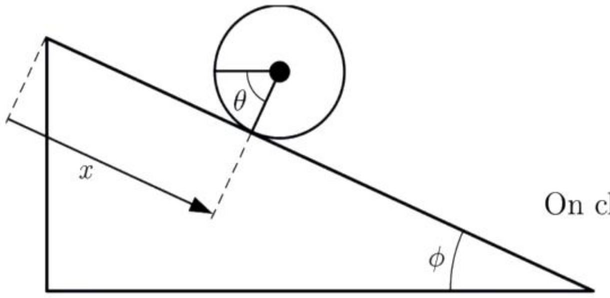
We will now use this principle to see how with Lagrange we can calculate constraint forces. Nothing beats an example. I take back my problem of the cylinder going down an inclined plane. I write my lagrangian in terms of the theta variable and the x variable. But instead of doing what I did last time, I'll impose the fact that there is a constraint, but I'll keep the two variables. That's how we're going to make a force appear, the constraint force, the constraint being, here, the rolling without slipping. So, we have, we are looking for the extremum of the action defined for this L under the constraint that x is R theta. So it's the same as considering the Lagrangian L prime, which is the Lagrangian I had, minus a certain coefficient; I called it F because I know where I'm going to go. But this, it's a coefficient called a Lagrange multiplier which multiplies this relation of constraint x minus R theta. And I consider this Lagrangian L prime now as having two independent variables. So, if I, now I write my equations of motion, I calculate d of L prime, the, the derivative of this lagrangian with respect to theta point. I calculate the derivative of L prime with respect to theta.

Notes

Summary

6m 13s

Application : forces de contraintes



$$L = \frac{1}{2} I_\Delta \dot{\theta}^2 + \frac{1}{2} M\dot{x}^2 + Mgx\sin\phi$$

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$$\frac{\partial L'}{\partial \dot{\theta}} = I_\Delta \dot{\theta} \qquad \frac{\partial L'}{\partial \theta} = FR \qquad I_\Delta \ddot{\theta} = FR$$ théorème du moment cinétique

$$\frac{\partial L'}{\partial \dot{x}} = M\dot{x} \qquad \frac{\partial L'}{\partial x} = Mg\sin\phi - F \qquad M\ddot{x} = Mg\sin\phi - F$$ théorème du centre de masse

So, we have FR which appears, and I have this equation. And well, I have a notation that is to recognize right away what is going on. We have here an equation of motion for the angular momentum. It's a little bit like applying d of Lg on dt equals external Mg with the momentum being worth the force, and here we have R, the radius. The second equation of motion, I get by taking x as a variable. So, I calculated d of L prime over d of x point, I calculate d of L prime over d of x, x appears here, and there. And what do I have here? I have this equation of motion which is the equation of motion, it's Newton's equation, the equation F equals Ma for the center of mass, with the force in the direction of the inclined plane. There is a MG sin alpha, and there is the frictional force. And now, I found everything that Newtonian mechanics could tell us, the angular momentum theorem, the center of mass theorem for this undeformable solid. Once again, I remind the method. We wrote the Lagrangian, and we have a constraint, and we add this constraint with a coefficient in front of it, which we call a Lagrange multiplier. And now, we write the Lagrange equations as if the variables were independent. This method is only valid for holonomic constraints.

Notes

Summary

8m 00s