ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Symétries et conservations

Mécanique, cours 26.1

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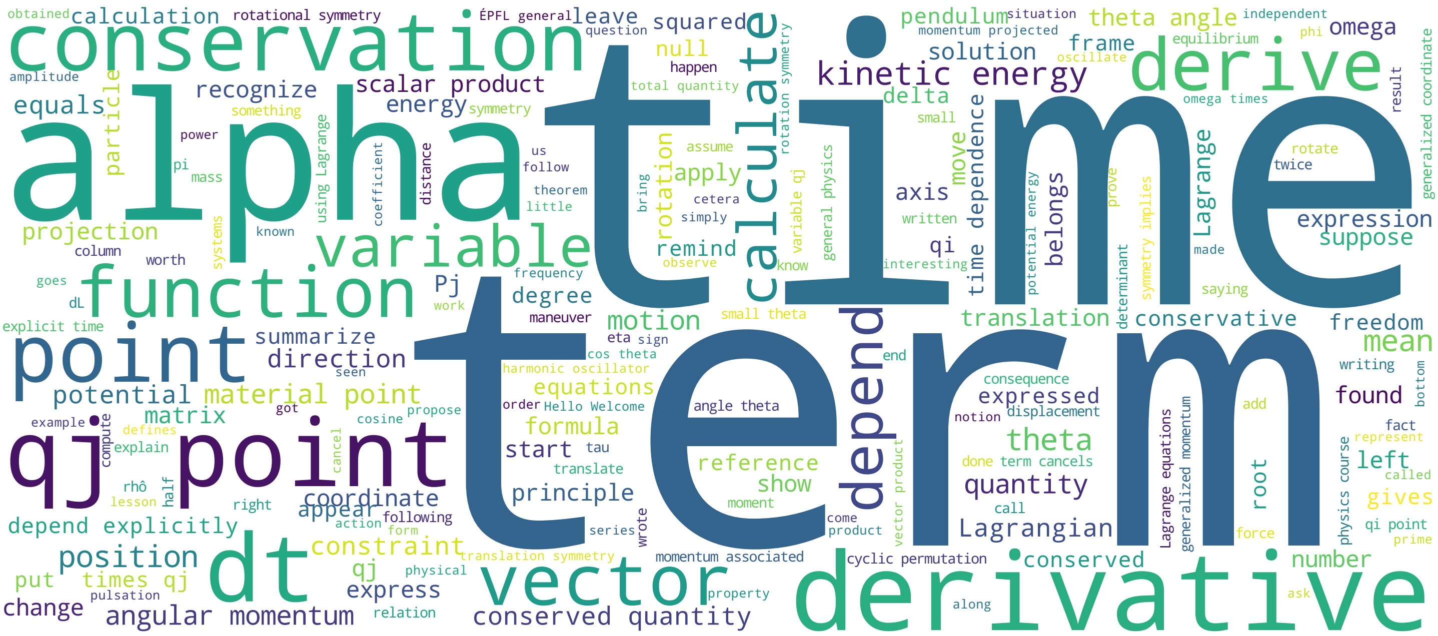

Video

EPFL

Symétries et conservations

- Variable cyclique
- Symétrie de translation
- Symétrie de rotation
- Energie

Hello. Welcome to the ÉPFL general physics course. In this lesson, I would like to use Lagrange's formalism to show you that conservation of momentum, or conservation of angular momentum, are properties that result from fundamental symmetry of the systems we consider. To begin with, I'll define a cyclic variable, and I'll show you that, a translational symmetry implies a conservation of momentum projected in the direction in which we can translate, then we'll see that a rotational symmetry implies a conservation of angular momentum, projected in, or on the axis of rotation, and then we'll see how to express the conservation of energy using Lagrange.

Notes

Summary

0m 03s

Définition : quantité de mouvement généralisée

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

p_j quantité de mouvement généralisée
associée à chaque coordonnée généralisée q_j

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I start with the notion of generalized momentum. If you have a physical system which is characterized by a Lagrangian L, for the variable j of the generalized coordinates, d of L on d of qj point, it's going to be what we call the moment, the generalized momentum, associated with that variable qj. Why do we bother to make this definition?

Notes

Summary

1m 07s

Propriété : variable cyclique et conservation

q_j ***cyclique*** si L ne dépend pas de q_j

$$\frac{\partial L}{\partial q_j} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{d}{dt} p_j$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

p_j constante du mouvement

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So we saw in the little examples of the previous lesson, that when we do this calculation, we can either find a quantity of motion, properly speaking, or a kinetic moment, but what is very interesting to see here, is the following property: a property of conservation, we say that if qj is cyclic, it means if qj doesn't depend, if L doesn't depend on qj, if we have that, then the momentum is conserved, indeed if d of L on d of qj is null, d of L on d of qj, by Lagrange, I can write it like that, in there, I have Pj, so I'm saying that d on dt, the derivative with respect to time of Pj is null, so it means that Pj is a conserved quantity. So you see, in a problem where you have a variable, one of the coordinates which does not appear in the expression of the Lagrangian, then you have a conservation. That's something very useful.

Notes

Summary

2m 07s

Symétrie de translation et conservation

Système de N points matériels en $\boldsymbol{x}_i$, $(i = 1, .., N)$

forces conservatives, potentiel $V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$

$\hat{\boldsymbol{n}}$ dans le référentiel tel que pour tout s :

$$V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = V\big(\boldsymbol{x}_1 + s\,\hat{\boldsymbol{n}}, \ldots, \boldsymbol{x}_N + s\,\hat{\boldsymbol{n}}\big)$$

$$0 = \frac{dL\big(\boldsymbol{x}_1 + s\,\hat{\boldsymbol{n}}, \ldots, \boldsymbol{x}_N + s\,\hat{\boldsymbol{n}}, \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N, t\big)}{ds}$$

Now, I'd like to get into, something a little more general, I'd like to consider systems of material points, with either translational or rotational symmetry. I start with translation. I suppose I have N material points, given by the positions, the vectors xi, i equals one to N, N is the number of material points, I don't have to worry here about reducing the number of variables, I'll suppose I have particles at the bottom which are free, I have a conservative system, so I have a potential, which defines my system, and I now assume that there is a direction of the frame of reference, such that if I make a displacement of the system in, along this direction, so each particle of the system I move it by s times n, it's a displacement in the direction of n, by a distance s, well I don't change the potential. So I have a translation symmetry for the potential. And we'll see that when we have that, we have conservation of momentum. So, I translate now the fact that L, if V doesn't change in this translation, then neither does L, because the kinetic energy obviously doesn't change in such a translation, and to express that L is independent of s, I write d of L over d of s, equals zero.

Notes

Summary

3m 02s

Symétrie de translation et conservation

Système de N points matériels en $\boldsymbol{x}_i$, $(i = 1, .., N)$

forces conservatives, potentiel $V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$

$\hat{\boldsymbol{n}}$ dans le référentiel tel que pour tout s :

$$V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = V\big(\boldsymbol{x}_1 + s\,\hat{\boldsymbol{n}}, \ldots, \boldsymbol{x}_N + s\,\hat{\boldsymbol{n}}\big)$$

$$0 = \frac{dL\big(\boldsymbol{x}_1 + s\,\hat{\boldsymbol{n}}, \ldots, \boldsymbol{x}_N + s\,\hat{\boldsymbol{n}}, \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N, t\big)}{ds} = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial x_{\alpha i}} n_i = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) n_i$$

$$= \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_{\alpha i}}\right) n_i = \frac{d}{dt}\left(\sum_{\alpha=1}^{N} m_\alpha\, \dot{\boldsymbol{x}}_\alpha\right) \cdot \hat{\boldsymbol{n}}$$

but here to explain this derivative, I remind that L is a function here of x one, x two, et cetera up to xN, but that here I didn't just take x one, but I made this translation of a distance s. And now I have a scalar function which is a function of s, but through all these variables, well these position vectors. So how do I do this derivative d on ds? I have to derive with respect to each position, or each coordinate of position, x alpha i, and then I have to derive the position alpha i, x alpha i with respect to s. So I have a derivative of that term with respect to s, which always gives me the ni with i which goes from one to three. This sum is the sum on the particles. Now, for this term, I use Lagrange, so this term, I say that it is worth d on dt, d of L, on d of x alpha i point, that, I recognize the Pi, the generalized momentum associated to the variable x alpha i, now, in L, there is only the kinetic energy which depends on the x alpha i point, so I can write, replace L by T, when I derive the kinetic energy with respect to x alpha i point, I end up with the following term, the term m alpha, x alpha point, index i times ni, I have here removed the sum over i, I have kept only the sum over alpha because I have here introduced the scalar product.

Notes

Summary

5m 03s

Symétrie de translation et conservation

Système de N points matériels en $\boldsymbol{x}_i$, $(i = 1, .., N)$
forces conservatives, potentiel $V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$
$\hat{\boldsymbol{n}}$ dans le référentiel tel que pour tout s :

$$V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = V\big(\boldsymbol{x}_1 + s\,\hat{\boldsymbol{n}}, \ldots, \boldsymbol{x}_N + s\,\hat{\boldsymbol{n}}\big)$$

$$0 = \frac{dL\big(\boldsymbol{x}_1 + s\,\hat{\boldsymbol{n}}, \ldots, \boldsymbol{x}_N + s\,\hat{\boldsymbol{n}}, \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N, t\big)}{ds} = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial L}{\partial x_{\alpha i}} n_i = \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) n_i$$

$$= \sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_{\alpha i}}\right) n_i = \frac{d}{dt}\left(\sum_{\alpha=1}^{N} m_\alpha\,\dot{\boldsymbol{x}}_\alpha\right)\cdot\hat{\boldsymbol{n}} \qquad \sum_{\alpha=1}^{N} m_\alpha\,\dot{\boldsymbol{x}}_\alpha = \boldsymbol{P} \text{ quantité de mouvement totale}$$

$\boldsymbol{P}\cdot\hat{\boldsymbol{n}}$ grandeur conservée

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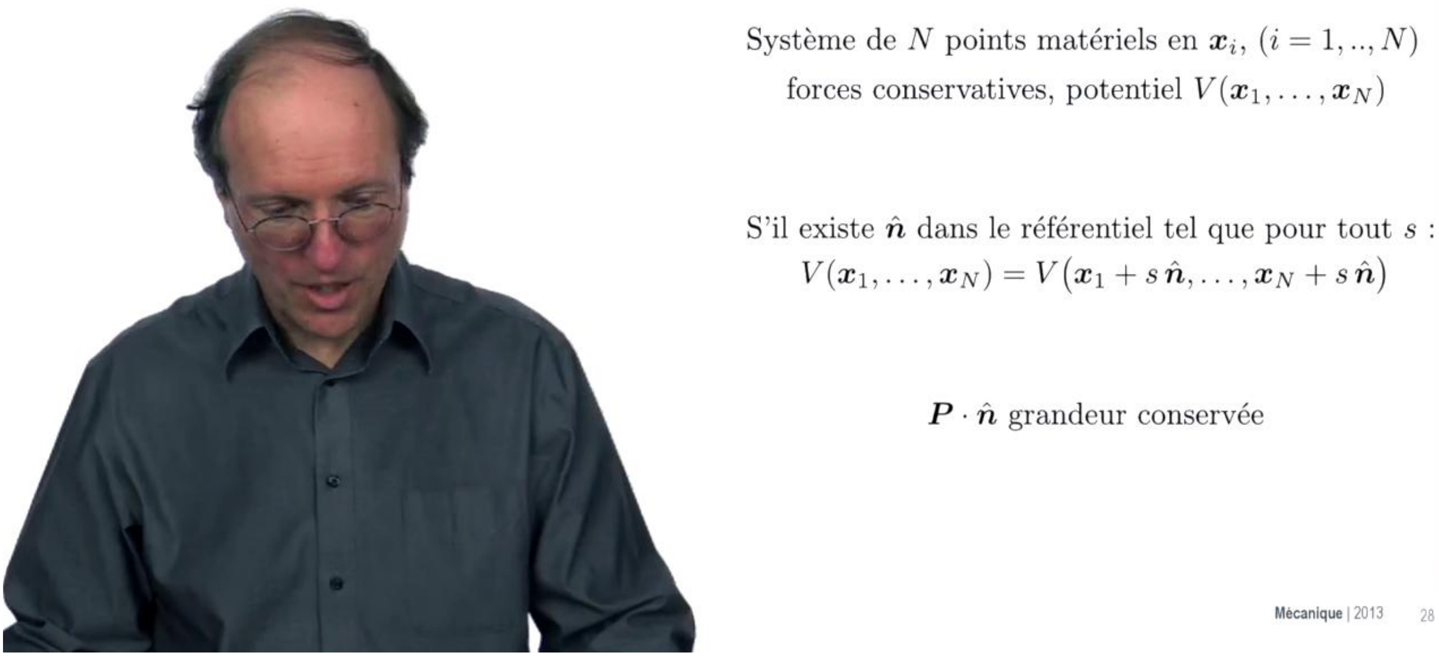
And now the d on dt, I can bring it on n, because n belongs to the frame of reference, so I'm going to add a zero, and at this moment, I'm saying that the derivative with respect to time of this scalar product with n, is zero, that means that this term, which is the total quantity of motion, times the unit vector n, so it's the projection of the quantity of motion in the direction n which is a conserved quantity, since its derivative with respect to time is zero.

Notes

Summary

7m 21s

Théorème de Noether : translation



I summarize, and what I have just expressed is known in the literature as Noether's theorem, the theorem of Mrs. Emma Noether. We have a system of N material points. Which is conservative, so all forces derive from a potential. And I have this translation property, for the potential. So, I obtained thanks to Lagrange that the total quantity of motion projected in this direction of translation, is a conserved quantity. So you see that here, the conservation of the quantity of motion results from a principle of symmetry. This is a very strong and elegant result.

Notes

Summary

8m 10s

Symétrie de rotation et conservation

$$V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = V(\boldsymbol{R}_\theta \boldsymbol{x}_1, \ldots, \boldsymbol{R}_\theta \boldsymbol{x}_N)$$

$\boldsymbol{R}_\theta$ axe passant par l'origine O, d'orientation $\hat{\boldsymbol{n}}$, d'angle θ

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \wedge \boldsymbol{r} \qquad \rightarrow \qquad \boldsymbol{R}_\theta \boldsymbol{x}_\alpha = \boldsymbol{x}_\alpha + \theta\, \hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha \qquad \theta \text{ petit}$$

$$0 = \frac{dL\big(\boldsymbol{x}_1 + \theta\, \hat{\boldsymbol{n}} \wedge \boldsymbol{x}_1, \ldots, \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N, t\big)}{d\theta}$$

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I now move to a rotation symmetry. Same situation, I have a conservative system, and now I suppose that if I rotate the system by an angle theta around an axis which passes through the origin, and the axis is an axis which belongs to the frame of reference, then I make this hypothesis that the potential does not change. And we will find in this case that we have a conservation of angular momentum. Now, I have to make a mathematical representation of this rotation of an angle theta. So we worked a lot on infinitesimal rotations. Let's look at what happens if we have a rotation of a small theta angle. I remind you of this well known formula which had led to the equations of, to the Poisson formulas, and I apply it for a dt. So, I have an omega times dt. I'm going to assume that I have a theta angle that is small, which is omega times dt. So I can apply this formula to say that the vector I get from the rotation of theta and that I have, a rotation that I apply to vector x alpha, is x alpha plus the term that follows from this formula, a theta n times vector product with x alpha. So now, to express this symmetry, I can say, the derivative of the Lagrangian with respect to theta, when I estimate the Lagrangian opposition x one increased by that term, up to xn.

Notes

Summary

9m 06s

$$V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = V(\boldsymbol{R}_\theta \boldsymbol{x}_1, \ldots, \boldsymbol{R}_\theta \boldsymbol{x}_N)$$

$\boldsymbol{R}_\theta$ axe passant par l'origine O, d'orientation $\hat{\boldsymbol{n}}$, d'angle θ

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \wedge \boldsymbol{r} \qquad \rightarrow \qquad \boldsymbol{R}_\theta \boldsymbol{x}_\alpha = \boldsymbol{x}_\alpha + \theta\, \hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha \qquad \theta \text{ petit}$$

$$0 = \frac{dL\big(\boldsymbol{x}_1 + \theta\, \hat{\boldsymbol{n}} \wedge \boldsymbol{x}_1, \ldots, \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N, t\big)}{d\theta} = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{\partial L}{\partial x_{\alpha i}} \left(\hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha\right)_i = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) \left(\hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha\right)_i$$

$$= \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \left(\frac{d\boldsymbol{p}_\alpha}{dt}\right)_i \left(\hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha\right)_i$$

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Then afterwards, there are still obviously the variables x a point xn point L time which is, is not excluded, this explicit time dependence. So, to do this calculation, L being a function of several variables, I have to derive with respect to each variable. So, this is what I wrote here, d of L over d of alpha x alpha i. And then I have to derive x alpha i with respect to theta. So, that's a term like that that I have to derive with respect to theta. That leaves me with n cross x alpha, component i. Now, I'm going to use Lagrange for that term. So, this term is equal to this term, it's the Lagrange equations. And here, I simply copied this last term. Here, I recognize a term that depends only on the kinetic energy. And when I derive, I get the quantity of motion, that's what I expressed here. We have the d over dt of the momentum, here it is. And this term again, sum over i equals one to three. And now I propose to myself, what do I have here? I have the scalar product of this vector here, and this one. And so, I have a mixed product that I can calculate by, like a determinant where I put this vector in the first column, this vector in the second, this vector in the third.

Notes

Summary

11m 03s

Symétrie de rotation et conservation

$$V(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = V(\boldsymbol{R}_\theta \boldsymbol{x}_1, \dots, \boldsymbol{R}_\theta \boldsymbol{x}_N)$$

$\boldsymbol{R}_\theta$ axe passant par l'origine O, d'orientation $\hat{\boldsymbol{n}}$, d'angle θ

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \wedge \boldsymbol{r} \qquad \rightarrow \qquad \boldsymbol{R}_\theta \boldsymbol{x}_\alpha = \boldsymbol{x}_\alpha + \theta\, \hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha \qquad \theta \text{ petit}$$

$$0 = \frac{dL\big(\boldsymbol{x}_1 + \theta\, \hat{\boldsymbol{n}} \wedge \boldsymbol{x}_1, \dots, \dot{\boldsymbol{x}}_1, \dots, \dot{\boldsymbol{x}}_N, t\big)}{d\theta} = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{\partial L}{\partial x_{\alpha i}} \left(\hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha\right)_i = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) \left(\hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha\right)_i$$

$$= \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \left(\frac{d\boldsymbol{p}_\alpha}{dt}\right)_i \left(\hat{\boldsymbol{n}} \wedge \boldsymbol{x}_\alpha\right)_i = \sum_{\alpha=1}^{N} \sum_{i=1}^{3} (\hat{\boldsymbol{n}})_i \left(\boldsymbol{x}_\alpha \wedge \frac{d\boldsymbol{p}_\alpha}{dt}\right)_i = \hat{\boldsymbol{n}} \cdot \sum_{\alpha=1}^{N} \left(\boldsymbol{x}_\alpha \wedge \frac{d\boldsymbol{p}_\alpha}{dt}\right)$$

$$= \hat{\boldsymbol{n}} \cdot \sum_{\alpha=1}^{N} \frac{d}{dt}\left(\boldsymbol{x}_\alpha \wedge \boldsymbol{p}_\alpha\right) = \frac{d}{dt}\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{L}_O\right) \qquad \boldsymbol{L}_O = \sum_{\alpha=1}^{N} \left(\boldsymbol{x}_\alpha \wedge \boldsymbol{p}_\alpha\right)$$

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Now, you know when you calculate a determinant, you can do cyclic permutations of the columns. Here, that means I can do a cyclic permutation of vectors. I'm going to move the n to this position, the x alpha to this position, and this vector, here. That's a cyclic permutation. It gives me this. There's the n, the x alpha of dp over dn, dt, dp alpha over dt. And now I do a little extra maneuvering. Okay, I'm doing two maneuvers. Okay, I'm just doing one more maneuver. Here, I just expressed this scalar product, here. And now, the d on dt, I can take it out of the parenthesis, because I add a term that is d on dt of the x alpha, but that is collinear with P alpha, the vector product gives zero. And since the n we're talking about here, belongs to the referential, I can again encompass it. The, therefore, I can take out d on dt totally. And I recognize here sum over alpha x alpha cross P alpha, this is our definition of angular momentum. And so, I found that the derivative with respect to time of the projection of angular momentum in the n direction is zero, that is, this projection is conserved.

Notes

Summary

12m 45s

Théorème de Noether : rotation

$$V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = V(\boldsymbol{R}_\theta \boldsymbol{x}_1, \ldots, \boldsymbol{R}_\theta \boldsymbol{x}_N)$$

$\boldsymbol{R}_\theta$ axe passant par l'origine,

d'orientation $\hat{\boldsymbol{n}}$, d'angle θ

$\hat{\boldsymbol{n}} \cdot \boldsymbol{L}_O$ conservée

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Now, I summarize. You have a material point system with conservative forces. You have a rotational symmetry of an n-axis that belongs to the reference frame. Well, we just found that nothing changes when you rotate by an infinitely small theta angle. If we make a series of rotations of small theta angles, nothing always changes, so we have a conservation, whatever the theta angle is. And so we have this relation between the conservation of the projection of angular momentum and the symmetry of the problem. So, we see that this, this principle of conservation of angular momentum results from a fundamental principle of symmetry. This is a very nice result.

Notes

Summary

14m 27s

Autre grandeur conservée

Si L ne dépend pas **explicitement** du temps $\mathcal{H} = \sum_{j=1}^{n} p_j \dot{q}_j - L$ est une grandeur conservée

$$\frac{d\mathcal{H}}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - \frac{dL}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\dot{q}_j$$

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Now, I move to the question of energy. So first of all, I'm going to find a conserved quantity which may seem a bit singular. I consider a system where the Lagrangian does not depend explicitly on time. And I'm going to show you now that this quantity, where pj, is the momentum associated to the variable qj. And I make a sum on j equal to one to n, the number of degrees of freedom of these products, I subtract the Lagrangian, and this quantity is a conserved quantity. Demonstration. To prove that H is a conserved quantity, I will calculate the derivative of H with respect to time, and we will show that this derivative is null. So we have to calculate the derivative with respect to time of this term, minus the derivative with respect to time, a right d dl on dt. So I'll work on that second term. L is a function of qi, of qi points, and in principle of time. We assumed that there was no time dependence. So L here depends on qi and qi point. So, to compute the derivative d on dt, d right, I have to compute the dL on d from q, j point, sum on j, d from qj point times qj point, and I have d from L on d from qj times qj point. I derive L with respect to each variable, and I derive each variable with respect to time.

Notes

Summary

15m 19s

Autre grandeur conservée

Si L ne dépend pas **explicitement** du temps $\mathcal{H} = \sum_{j=1}^{n} p_j \dot{q}_j - L$ est une grandeur conservée

$$\frac{d\mathcal{H}}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - \frac{dL}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\dot{q}_j$$

$$= \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j = 0$$

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Now, in the first, these, these first two terms, I leave them. and this term here, I'm going to write it as d over dt, well, I'm using Lagrange, and I'm writing it by Lagrange as d over dt of d from L over d from qj point. And I'm done, because if now I want to write this derivative explicitly, I'll have a derivative with respect to time, here, which will therefore be equivalent, maybe it's better to write it. If I want to develop this term, I should write sum on j of d on dt, of d of L on d of qj point times qj point. That's a term. If I now carry the derivative over to the other term, I have sum over j, d from L over d from qj point, qj point point. This term cancels out with this one, and this term cancels out with this one. So, I'm left with zero. I clean up, that's it. So I found that the derivative with respect to time of this quantity H is zero. That means that H is a conserved quantity. The good question to ask is but what does H represent? Then I have the following result.

Notes

Summary

17m 13s

Conservation de l'énergie mécanique

Si L ne dépend pas **explicitement** du temps

Si les contraintes ne dépendent pas du temps

$$\mathcal{H} = \sum_{j=1}^{n} p_j \dot{q}_j - L$$

grandeur conservée, est l'énergie mécanique

$$\sum_{j=1}^{n} p_j \dot{q}_j = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = \sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j$$

En général, pour un point matériel : $\boldsymbol{v} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}}{\partial t} \implies T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \left(\sum_{k=i}^{n} \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\dot{q}_k\right)^2$

$$\sum_{j=1}^{n} p_j \dot{q}_j = \sum_{\alpha=1}^{N} m_\alpha \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\dot{q}_k\right) \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}\dot{q}_j = 2T$$

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If L does not depend explicitly on time, and if the constraints do not depend on time, then H, which is a conserved quantity, is the mechanical energy. Let's prove it. I start from this term. I explain the pj by their definition. d of L on d of qj point can only be d of t on d of qj point, because v cannot depend on the qj points. Now, in all generality, we have seen that v can be written like this. If now, the constraints do not depend on time, this term falls. I only have this term left for v which I have rewritten here. So, in the expression of the kinetic energy, well the expression of the kinetic energy can be done like this, only if we don't have this term. Now, if we have it, then we see that when we calculate pj, qj point, we can write it like this. And that, so that means I have to derive that with respect to qj. So, there is half a time the two that comes from here, that has, cancels each other. I'm left with one time that sum that's there, and after the, the, the particular point, d of r alpha on d of qj times qj point, when I drift with respect to the variable qj point. I can rearrange these terms. This sum, it only relates to this.

Notes

Summary

18m 48s

Conservation de l'énergie mécanique

Si L ne dépend pas **explicitement** du temps

Si les contraintes ne dépendent pas du temps

$$\mathcal{H} = \sum_{j=1}^{n} p_j \dot{q}_j - L$$

grandeur conservée, est l'énergie mécanique

$$\sum_{j=1}^{n} p_j \dot{q}_j = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = \sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j$$

En général, pour un point matériel : $\boldsymbol{v} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}}{\partial t} \implies T = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha \left(\sum_{k=i}^{n} \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k} \dot{q}_k \right)^2$

$$\sum_{j=1}^{n} p_j \dot{q}_j = \sum_{\alpha=1}^{N} m_\alpha \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k} \dot{q}_k \right) \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \dot{q}_j = 2T$$

$$\mathcal{H} = 2T - L = T + V$$

Mécanique | 2013 62

So we have this sum times the same sum, once indexed on k, but m times on j, but it is the same sum. So, we see that we have two times, we have a v-squared, we have m alpha, v alpha squared. So, we have twice the kinetic energy. So now, H is this term, two T minus L. But L is T minus V, so that leaves T plus V. And this is the energy, so we found that if L does not depend explicitly on time, this quantity is conserved. If the constraints do not depend on time, then this term is the energy, and we have therefore that the energy is conserved.

Notes

Summary

21m 02s

- Conservation de la quantité de mouvement
- Conservation du moment cinétique
- Conservation de l'énergie

Mécanique | 2013 66

I summarize. We have obtained a conservation of the quantity of motion which is a consequence of a translation symmetry. The conservation of angular momentum which is a consequence of a rotation symmetry. And then, we got the conservation of energy, if we have a system where we have no explicit time dependence, and the constraint does not depend explicitly on time.

Notes

Summary

21m 49s