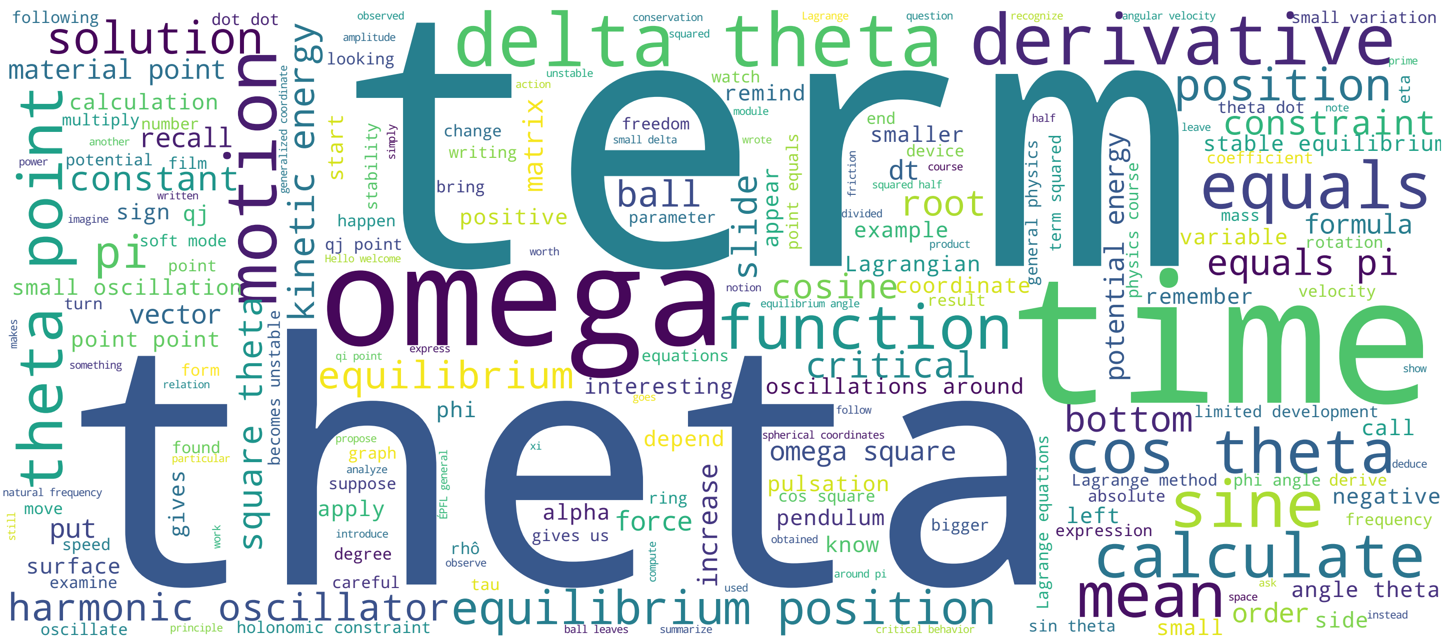
ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Expériences : contraintes holonomes

Mécanique, cours 25.exp

Jean-Philippe Ansermet



Video

EPFL

Expériences : contraintes holonomes

- Contrainte indépendante du temps
- Contrainte dépendante du temps
- Stabilité d'un équilibre
- Petites oscillations autour d'un équilibre stable
- Mode « mou »

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Hello, welcome to the ÉPFL general physics course. In this module, we will look concretely at some examples of holonomic constraints. First, we will look at a time independent constraint, then a time dependent constraint, and after that, I would like to discuss on an example using Lagrange equations, the question of equilibrium stabilities, and small oscillations around an equilibrium. The example I chose here is of particular interest to me because we arrive at what I call a soft mode, you will see, a vibratory mode whose frequency tends towards zero when we change a parameter of the system.

Notes

Summary

0m 03s

Contrainte indépendante du temps

- Contrainte indépendante du temps : point matériel astreint à se déplacer sur une surface dans le référentiel.

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I start with a very simple constraint, see this surface, you imagine a ball forced to move on this surface, the equation of the surface, it's one of these constraint equations, called holonomic constraint.

Notes

Summary

0m 54s

Contrainte dépendante du temps

- Contrainte dépendante du temps : point matériel astreint à se déplacer sur un cercle en rotation uniforme d'axe vertical.
- Le déplacement virtuel est le long de l'anneau, différent du déplacement réel pendant un temps donné.

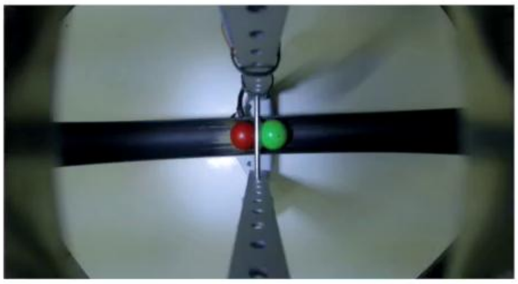
On the other hand, if you take the problem of the ball in the ring, you remember that, we have balls rolling in this hemispherical slide, and the slide itself rotates around a vertical axis, then there, we have a constraint which depends on time. It's the phi angle that we used to locate the position of the ball in the ring, the phi angle which is the azimuthal angle to the ground on the white board, which is a function of time and is equal to omega t, with omega which is constant.

Notes

Summary

1m 14s

Equilibre et petites oscillations



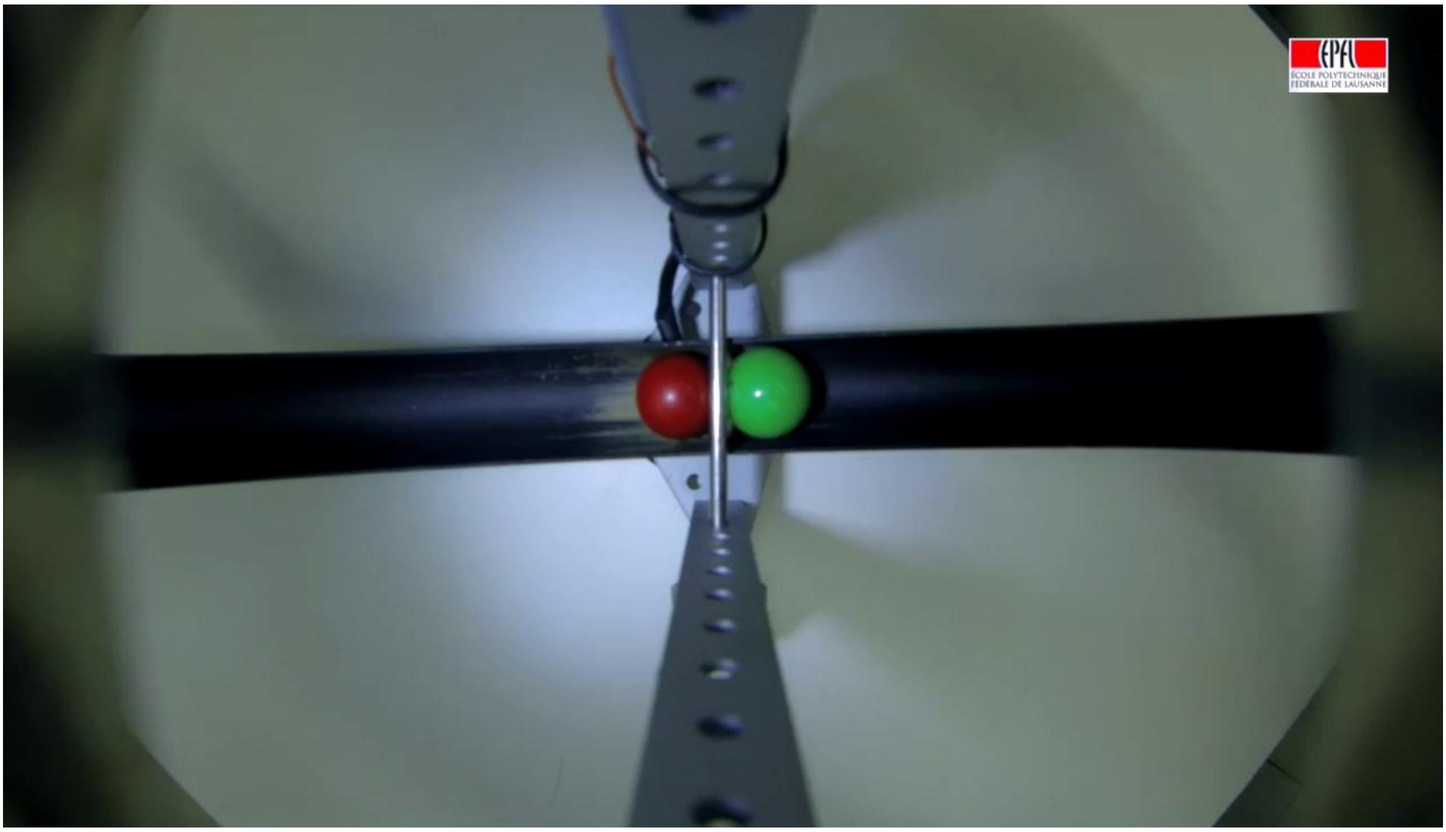
- Anneau en rotation : la bille oscille autour d'une position d'équilibre. Quelle est cette position ?
- Quelle est cette fréquence ?
- Pratiquement, les frottements font que la bille s'immobilise rapidement à sa position d'équilibre.

What I would like to do here is to examine the equilibrium of the ball, and then the small oscillations around the equilibrium. Before we get into the calculations, to remind ourselves of this physical system, I suggest that you watch a film, I, on this film, the angular velocity of the platform increases nicely, up to the point where the ball leaves its equilibrium at the bottom, and what I ask you to note is that as it leaves, this position at the bottom, it oscillates. And what we want to determine in the calculation that will follow, is the frequency of the small oscillations around the equilibrium position. On this device, there is friction, and the oscillation does not last long. I invite you to watch.

Notes

Summary

1m 54s



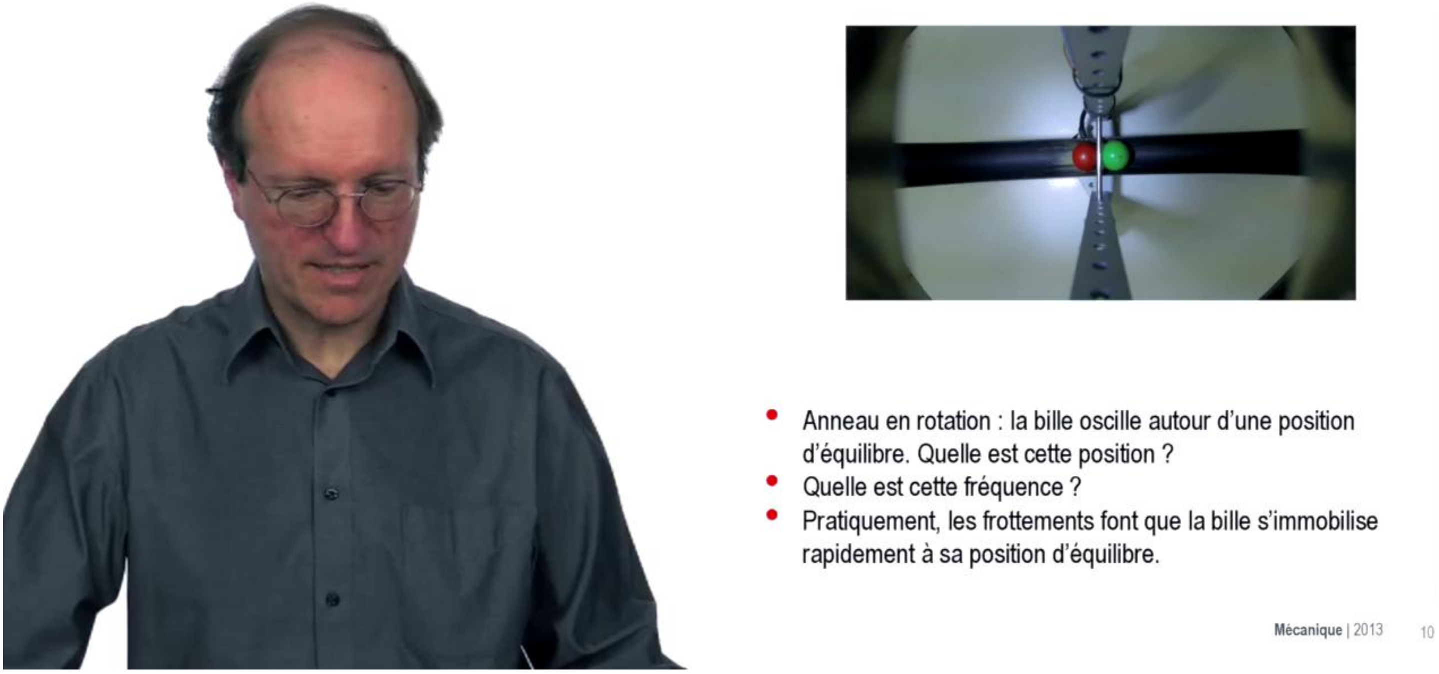
Here, we increase the speed of rotation, this equilibrium becomes unstable and here is the ball which oscillates sideways. The camera is obviously mounted on the rotating slide. And it is the bottom that seems to turn. Here are these oscillations. I stop the device, the ball goes back to the bottom.

Notes

Summary

2m 48s

Equilibre et petites oscillations



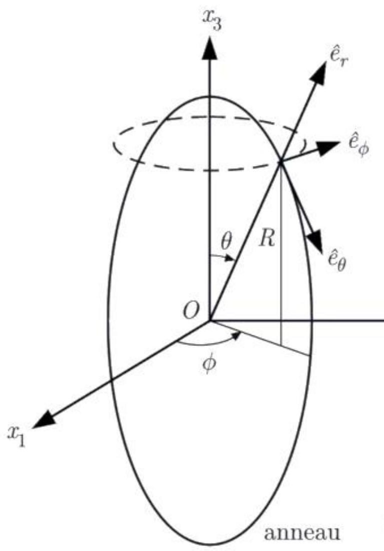
These are the oscillations around the lateral equilibrium position, that I would like to analyse.

Notes

Summary

3m 21s

Méthode de Lagrange



Vitesse : $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\dot{\phi}\sin\theta\boldsymbol{e}_\phi$

Contraintes : $r = R \qquad \phi = \omega t$

Energie cinétique : $T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta$

Energie potentielle : $V = mgR\cos\theta$

Lagrangien : $L = T - V$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mR^2\ddot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

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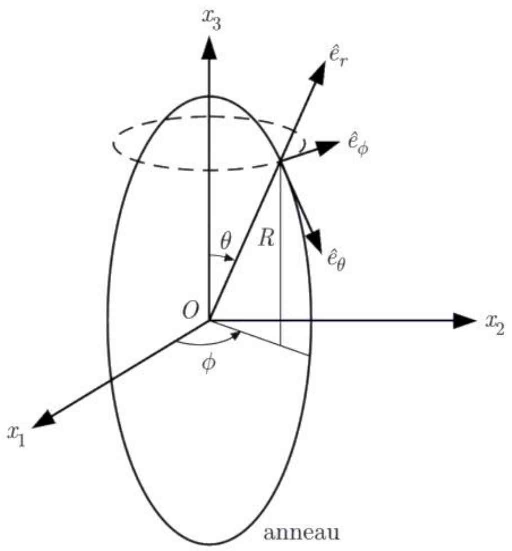
Of course, we'll do it with Lagrange's method. I propose to use, as last time we visited this problem, the spherical coordinates phi and theta, this is my slide, and, to work with acute angles I put the material point there up, so down there it is an angle theta equals pi, note well, and now I am going to focus on expressing the velocity, and the kinetic energy, to use the Lagrange method. So, the velocity in spherical coordinates, we know it. We have as constraint that the coordinate R is a constant, and phi, this rotation angle, is omega t, with omega being a constant. The kinetic energy will contain this term squared, half of m, R squared theta point squared, and then there is this term squared, half of m, R squared, omega squared sin squared theta. We'll take the potential energy mg times the height above the plane O x one, x two, that makes mgr cos theta. The Lagrangian is T minus V, I saved myself writing T minus V, maybe it's not ideal, I don't recommend it, but here for reason of space on the page, I didn't do it, we have to compute d of L on d of delta point, eh, I remind you Lagrange, d on dt, d of L on d of qi point, minus d of L on d of qi equals zero. That's what you have to apply.

Notes

Summary

3m 26s

Méthode de Lagrange



Vitesse : $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\dot{\phi}\sin\theta\boldsymbol{e}_\phi$

Contraintes : $r = R \qquad \phi = \omega t$

Energie cinétique : $T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta$

Energie potentielle : $V = mgR\cos\theta$

Lagrangien : $L = T - V$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mR^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta$$

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

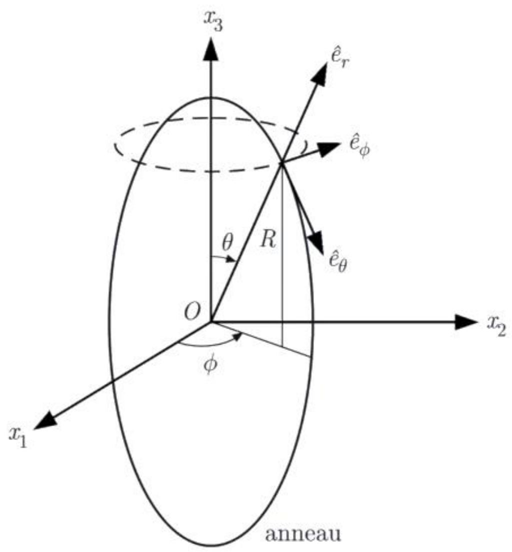
Then we calculate d of L over d of theta point, there's one here, that simply gives me mR square, theta point, and that I drift with respect to time. It just makes a theta point point, which happens. I have to calculate d of L over d of theta, so this is interesting because there is a theta term in the kinetic energy, and another in the potential energy. This one gives me sin theta cos theta, in the potential energy, it's minus V that we have to take, and the derivative of the cosine makes minus sine, so we'll have a plus to the sine here. And now, this, minus this equals zero, that's what we write like this, so we've already got the equation of motion.

Notes

Summary

5m 34s

Equilibres relatifs



$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

équilibre relatif : $\ddot{\theta} = 0$

$$\sin\theta_e\cos\theta_e = -\frac{g}{R\omega^2}\sin\theta_e$$

Trois solutions : $\theta_e = 0; \quad \theta_e = \pi; \quad \cos\theta_e = \frac{-g}{\omega^2 R}$

$$\cos\theta_e < 0 \implies \frac{\pi}{2} \le \theta_e \le \pi$$

$$|\cos\theta_e| < 1 \implies \omega \ge \omega_c = \sqrt{g/R}$$

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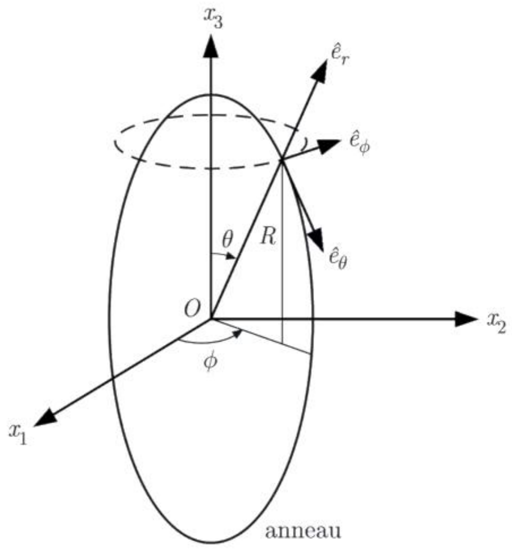
And now what we're interested in, is to analyze the equilibrium. First of all. I recall the equation of motion. We are looking for relative equilibrium, that means, we are looking at positions of the ball such that this angle theta doesn't move, well doesn't change, so we want theta point point equals zero in the equation of motion, so we have this equation, which admits several solutions. We can have sin theta e, which is zero, which means that, either theta e equals zero, or theta e equals pi, zero means that we're up there, we can be down there, or we can still have if sinus theta is not zero, then what we must have is cos theta equals minus g on omega kR Omega square R. We notice two things, the first one is that this term is negative, so I have cos theta e which is negative, that means theta e between pi on two and pi, pi on two would be there, and pi, so we are in that dial, when cos theta e is negative, and then we have to be careful in this formula, it's that the absolute value of the cosine is always smaller than one. That means that the absolute value of this fraction must be smaller than one.

Notes

Summary

6m 27s

Equilibres relatifs



$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

$$\text{équilibre relatif : } \ddot{\theta} = 0$$

$$\sin\theta_e \cos\theta_e = -\frac{g}{R\omega^2}\sin\theta_e$$

$$\text{Trois solutions : } \theta_e = 0; \quad \theta_e = \pi; \quad \cos\theta_e = \frac{-g}{\omega^2 R}$$

$$\cos\theta_e < 0 \implies \frac{\pi}{2} \leq \theta_e \leq \pi$$

$$|\cos\theta_e| < 1 \implies \omega \geq \omega_c = \sqrt{g/R}$$

Mécanique | 2013 27

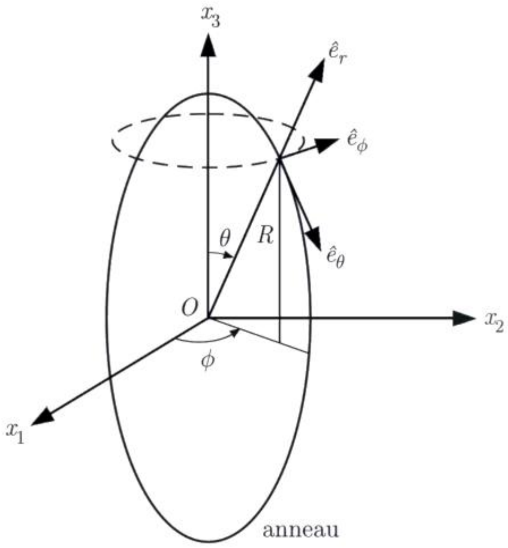
So we'll say that omega must be bigger, if it's smaller than one, it means that omega is bigger than a certain omega c for criticism, we'll see that this value intervenes everywhere, it's the root of g on R. So we have two equilibrium positions here and if omega is big enough we have a third position which appears here.

Notes

Summary

8m 04s

Stabilité des équilibres



Petites oscillations autour de l'équilibre : $\theta = \theta_e + \delta\theta$

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

équation dynamique au 1er ordre en $\delta\theta \ll 1$

$$\theta_e = 0 \qquad \ddot{\delta\theta} = \left(\omega^2 + \frac{g}{R}\right)\delta\theta$$

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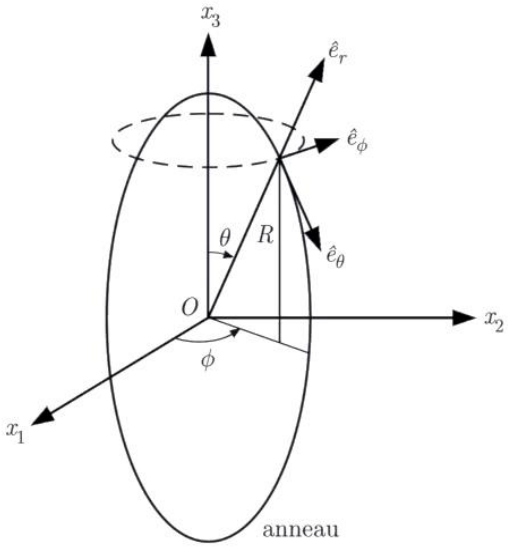
So now we're going to look at the stability of equilibria. What we're going to do is we're going to suppose that, with respect to an equilibrium position, we create a small deviation, and we're going to look at whether there will be a force that will bring the angle back to the equilibrium position, or there is a force that rather tends to go even further out of equilibrium. So we make a small variation of the equilibrium angle, which we call, which we note delta theta, and we'll look at what the equations of motion tell us. Here, I recall the equation of motion, and we'll take a small delta theta. So I start with the theta position e equals zero, that's up there. So we, by our intuition we already know the answer, this equilibrium is always unstable, so we'll check it. I'll write, instead of theta dot dot, I have delta theta dot dot, and then when I do these terms on the other side of the equals sign, then I have the sine theta which is about theta, the cos theta which is about one, and here I have a sine theta which is theta, I put the terms together, and I have this. Omega square is positive, g on R is positive, This coefficient of theta is positive, so we have an equation of motion that tells us that del theta is only going to increase.

Notes

Summary

8m 32s

Stabilité des équilibres



Petites oscillations autour de l'équilibre : $\theta = \theta_e + \delta\theta$

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

équation dynamique au 1er ordre en $\delta\theta \ll 1$

$\theta_e = 0 \qquad \delta\ddot{\theta} = \left(\omega^2 + \frac{g}{R}\right)\delta\theta \qquad$ toujours instable

$\theta_e = \pi$

$\theta = \pi + \delta\theta \qquad \cos(\pi + \delta\theta) \approx -1 \qquad \sin(\pi + \delta\theta) \approx -\delta\theta$

$\delta\ddot{\theta} = -\left(\frac{g}{R} - \omega^2\right)\delta\theta \qquad$ stable si $|\omega| < \omega_c$

instable si $|\omega| > \omega_c$

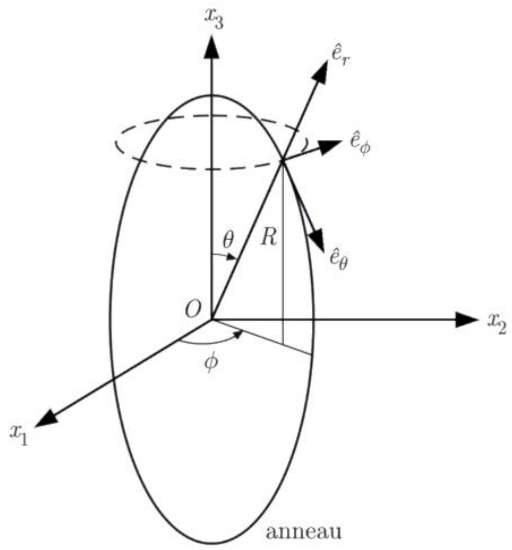
So it's an unstable equilibrium. Now let's take theta e equals pi, the second solution we had found, theta e equals pi, it's down there. Then we could expect that there, we have a stable equilibrium. Anyway, at low speed, we should find a stable equilibrium. Let's do the math, now we write that theta is theta e, plus a small variation, that's the general expression we have here for the particular case where theta e is pi. The cosine around pi, we'll take it equal to minus one, and the sine well if you look at your sine function around pi, you have less delta theta. The equation of motion, by putting these limited developments, takes the form here. So now we see that if omega is small, and in fact omega is smaller than the root of g on R, that's the critical value, here we have minus a positive term, so we have a harmonic oscillator equation, and we have a stable equilibrium, if now omega square is larger than g on R, this becomes positive and it's unstable. This corresponds to what we observed when we increase the angular velocity, so the phi point, we end up finding that the ball doesn't stay up there, it goes sideways. So I repeat, we have a harmonic oscillator as long as omega is small enough, and if omega is above the critical value, this balance becomes unstable.

Notes

Summary

10m 12s

Stabilité des équilibres



Petites oscillations autour de l'équilibre : $\theta = \theta_e + \delta\theta$

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0 \qquad \cos\theta_e = \frac{-g}{\omega^2 R}$$

équation dynamique au 1er ordre en $\delta\theta \ll 1$

$$\ddot{\delta\theta} - \omega^2 \sin(\theta_e + \delta\theta)\cos(\theta_e + \delta\theta) = \frac{g}{R}\sin(\theta_e + \delta\theta)$$

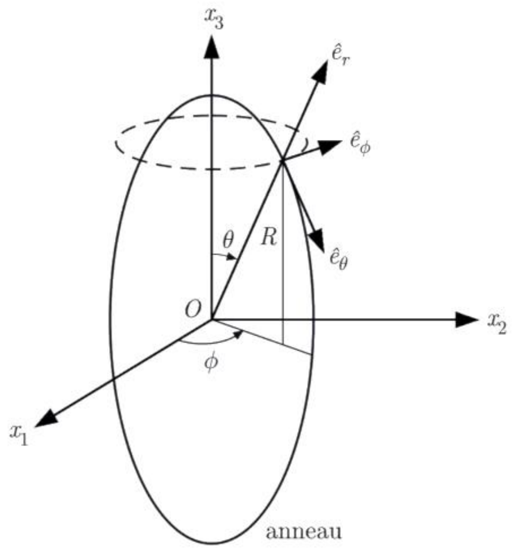
$$\ddot{\delta\theta} - \omega^2\left[\sin(\theta_e) + \delta\theta\cos(\theta_e)\right]\left[\cos(\theta_e) - \delta\theta\sin(\theta_e)\right] = \frac{g}{R}\left[\sin(\theta_e) + \delta\theta\cos(\theta_e)\right]$$

It still remains to examine the stability of the equilibrium around theta e, which is given by cos theta e equals minus g over omega square R, so I recall the equation of motion. I recall the equation that gives us theta e, we do a limited development for a small delta theta, so here we have a sine of theta e plus delta theta, for theta, I also write theta e plus delta theta, and there the theta sine I passed it to the other side of the equal sign, also theta e plus delta theta. And now I apply the formula of the limited development, for these three functions. So I've already done it, you know that, to the first order, we have delta theta times the derivative, the derivative of the sine and the cosine, the derivative of the cosine is minus the sine, and here we also have the cosine. So here we have two terms that multiply two terms. There will be four terms. But be careful, before writing everything down, remember that at order zero, we have so this term and this one, and this one on the other side of the equal sign. It is exactly these three terms that allowed us to find the equilibrium position. That was what we had when delta theta point point equals zero.

Notes

Summary

12m 22s



Petites oscillations autour de l'équilibre : $\theta = \theta_e + \delta\theta$

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0 \qquad \cos\theta_e = \frac{-g}{\omega^2 R}$$

équation dynamique au 1er ordre en $\delta\theta \ll 1$

$$\ddot{\delta\theta} - \omega^2 \sin(\theta_e + \delta\theta)\cos(\theta_e + \delta\theta) = \frac{g}{R}\sin(\theta_e + \delta\theta)$$

$$\ddot{\delta\theta} - \omega^2\left[\sin(\theta_e) + \delta\theta\cos(\theta_e)\right]\left[\cos(\theta_e) - \delta\theta\sin(\theta_e)\right] = \frac{g}{R}\left[\sin(\theta_e) + \delta\theta\cos(\theta_e)\right]$$

$$\ddot{\delta\theta} - \omega^2\,\delta\theta\left[-\sin^2(\theta_e) + \cos^2(\theta_e)\right] = \frac{g}{R}\,\delta\theta\cos(\theta_e)$$

$$\ddot{\delta\theta} = -\omega^2\sin^2(\theta_e)\,\delta\theta$$

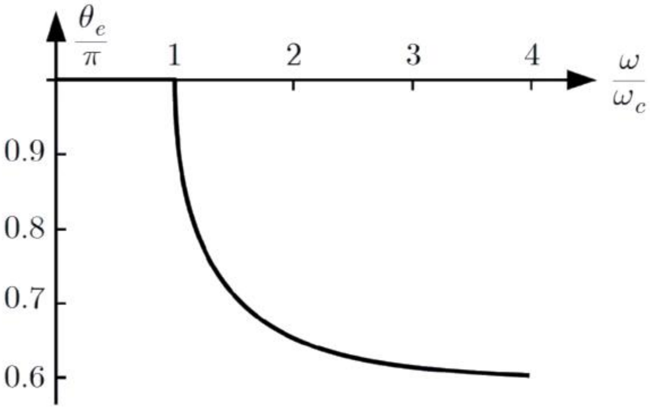
So, by definition of theta e, this term times this term, combined with this term, cancels. Now we have the second order terms. When I multiply this term by this term, I have a delta theta square term, we neglect them. So I'm left with this: with a cos square theta e, a square theta e sign. Now cos square theta e, or if you want the square sign theta e we can deduce it from this, with elementary trigonometry. And when I do the calculation, and I use for these theta e this term, it turns out that everything simplifies, everything cleans up, and only this is left. So here we have, this is a constant, so here we have once again the equation of a harmonic oscillator for the variable delta theta, but we have to remember that this solution only exists if this expression has a meaning. And we discussed the conditions, the notorious condition was that omega is above the critical value root of gi on R. And when this solution exists, then we have a harmonic oscillator, and this term gives us the squared pulsation of the harmonic oscillator. Well, I summarize the situation with a graph, which is the most interesting one, and this is where I introduce the notion of soft mode.

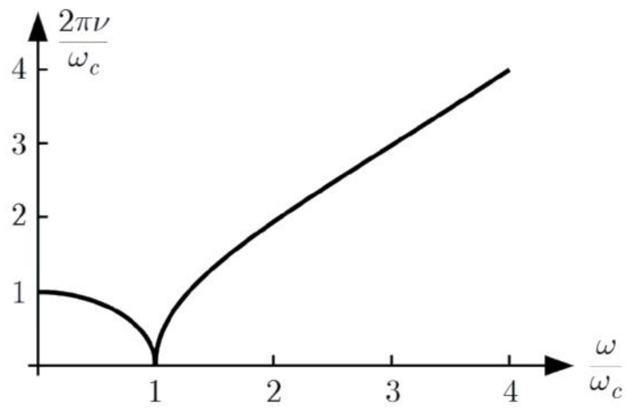
Notes

Summary

13m 59s

Mode “mou” de la bille dans l’anneau en rotation





I make a graph here of the equilibrium angle theta e divided by pi, so when we're at the bottom of the slide, at the bottom, we're at one, theta e equals pi, and if we're here I have the omega axis normalized by the critical omega value c, what we found, is that the stable equilibrium was at theta e equals pi, up to a critical value. And from there, hop, the ball leaves with an angle smaller than pi, so it goes up in the slide. And what is very interesting is that if I bring here the results we obtained for the pulsation, two pi nu, that is the pulsation of the harmonic oscillator that we found near the equilibrium positions, I normalizes it by the same omega c, so what I find, is that when we increase the speed of rotation of the slide, this frequency where the pulsation of the small oscillations decreases, at the critical value of omega, it becomes zero, and then it increases again. There are many physical situations where you have something like that, that is a, what is called a soft mode, an oscillatory system, which goes through a critical behavior, and in the vicinity of the critical behavior, the natural frequency of the oscillations becomes very small.

Notes

Summary

15m 45s