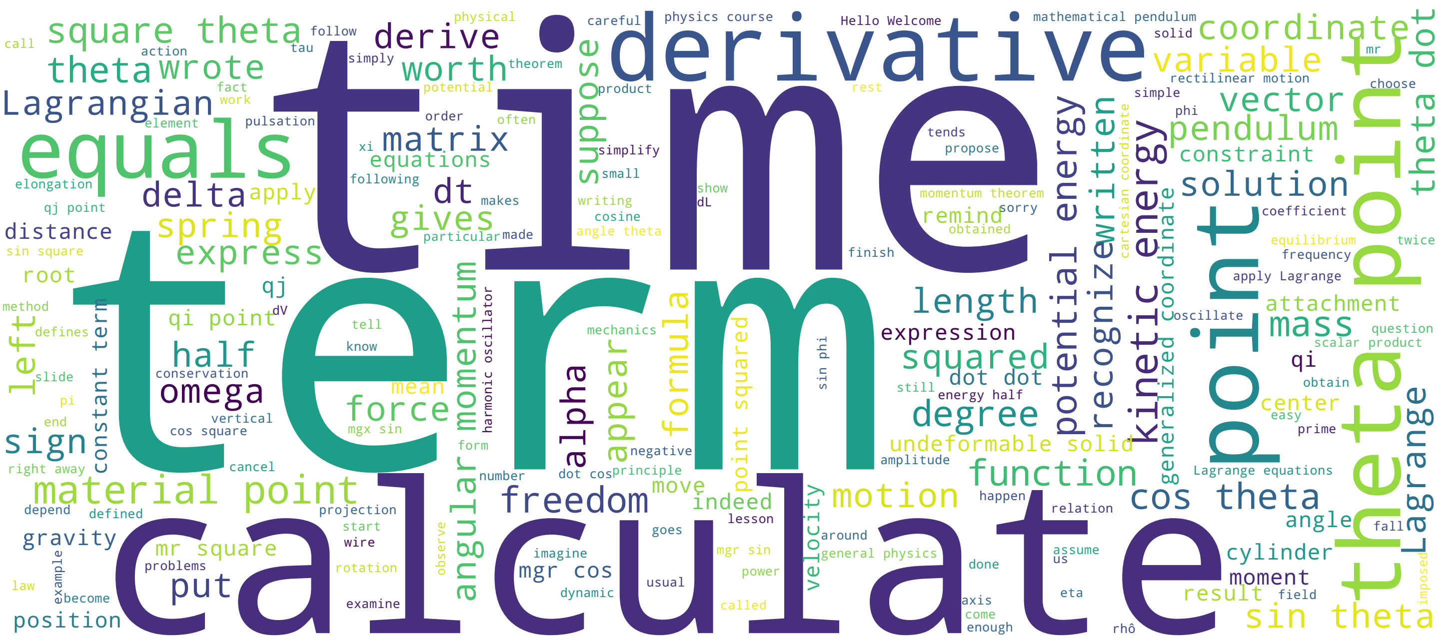
http://go.epfl.ch/traite-meca-4-3

Méthode de Lagrange, applications

Mécanique, cours 25.2

Jean-Philippe Ansermet



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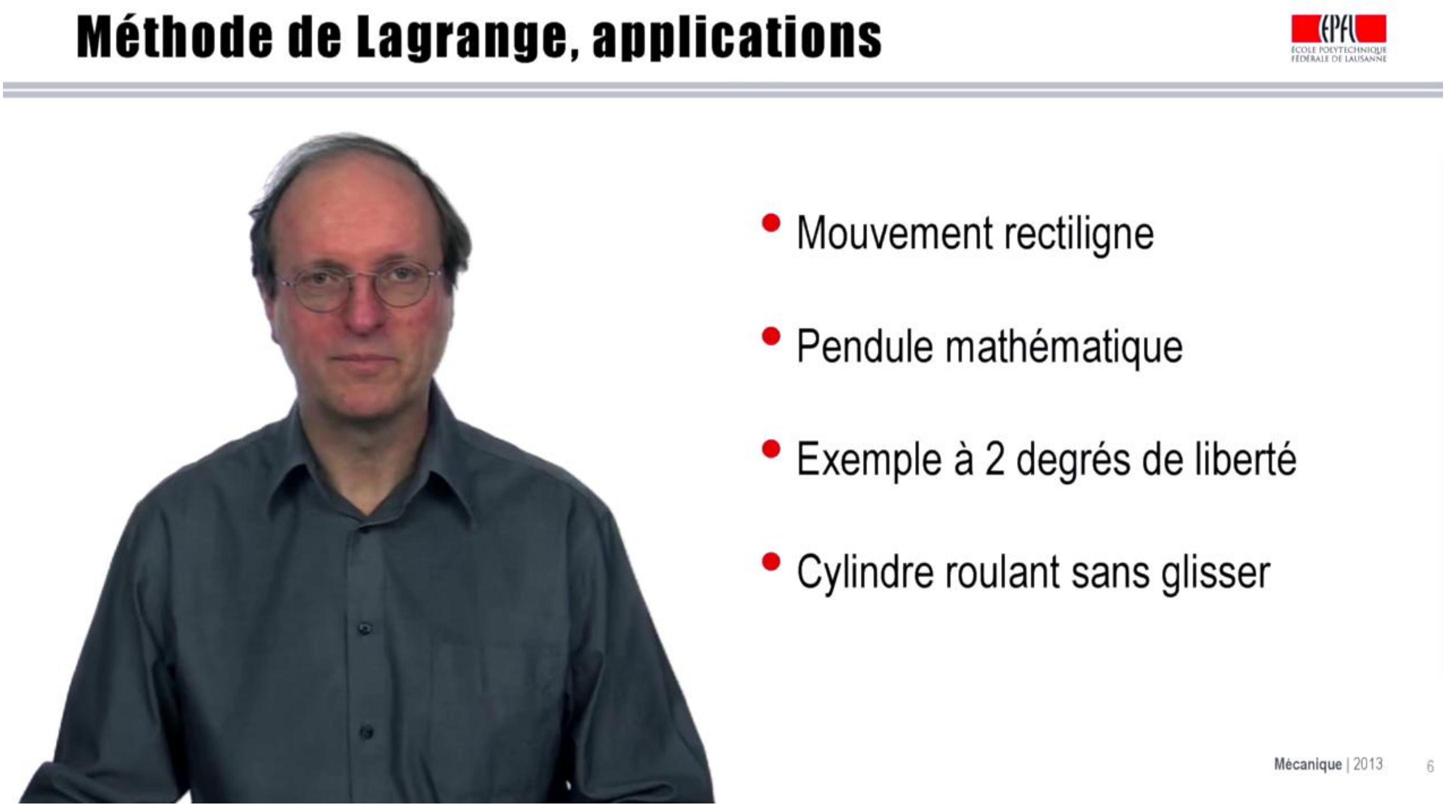


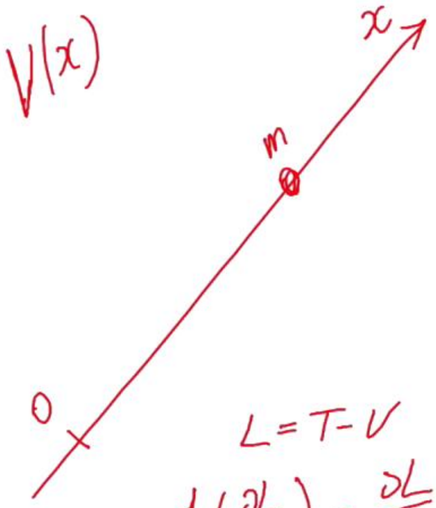
Hello. Welcome to the FPS General Physics course. In this lesson, I will show how to apply Lagrange's equations to problems of material points or undeformable solids. I'll first consider a rectilinear motion, then I'll look at the mathematical pendulum, so a motion that is defined by an angle, then I'll look at a problem with two degrees of freedom, and we'll finish with a problem of mechanics of the undeformable solid.

Notes

Summary

0m 03s

Mouvement rectiligne, force conservative



$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

I start with rectilinear motion. You have to imagine a material point forced to move on an axis. Then I will, as usual, give myself a cartesian axis which is carried by the line on which the material point moves, and I will designate by x the coordinate of this material point. It is very easy to obtain the kinetic energy. I remind you that we have L which is T minus V, and what we have to calculate is d over dt of d of L over d of qi point which is equal to d of L over d of qi. So we have to calculate L, we have to first calculate the kinetic energy and the potential energy. So for the potential energy, we will assume that we have any V of x that is given.

Notes

Summary

0m 38s

Mouvement rectiligne, force conservative

$$L = \frac{1}{2} m\dot{x}^2 - V(x)$$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$ la quantité de mouvement

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F$$ la force

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = m\ddot{x} = F$$

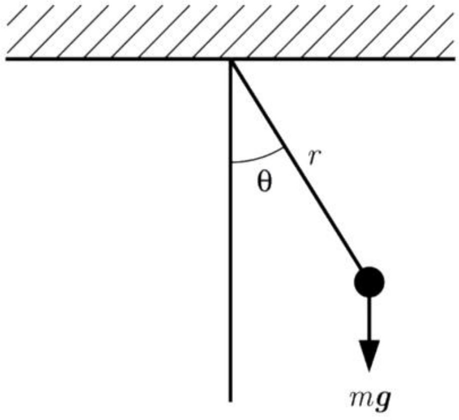
Then we have L which is worth the kinetic energy, half of mx squared point, minus the potential energy V of x. And we have to calculate d of L on dq point, here, it's d of L on d of x point, it's mx point, and we see right away that we have here the momentum. We have to calculate d of L on d of q, here, it is d of L on d of x. Here, I refer to the general notation. Here, it is for this particular case, my coordinate q, it is x, the cartesian coordinate of the material point. And L was minus V, so I have minus dV on dx. And I have, recognize here, the force. And now, the Lagrange equation tells me d on dt of that term, is equal to that. Well, we see that we are writing mx dot dot equals F. We simply wrote the law F equals ma for this problem. Now let's move on to a mathematical pendulum.

Notes

Summary

1m 56s

Pendule mathématique



$$L = T - V = \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

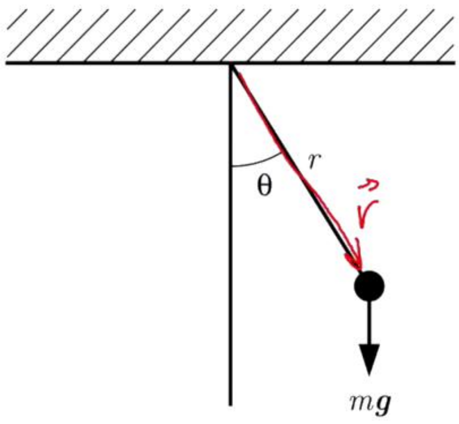
So here you have this straight line that defines the vertical, a wire with no mass, a mass at the end of mass m, in the field of gravity, a force mg. And I'll just calculate L equals T minus V. For the kinetic energy, we will have to consider the velocity. So the velocity, it is given in modulus by r theta point. And therefore, if I, I express my lagrangian L T minus V, I have here a half of mr square theta point. For the potential energy, well, I have to calculate if I put the zero of the potential energy when the material point is at its lowest, what I need is this distance there which is worth r minus r cos theta. And the potential energy V will be mgr un minus cos theta. Now I have a constant term here. What do we do with our Lagrangian? We're going to calculate derivatives of it, so the constant terms, they're useless to us. I'll take it out. And L, it was minus V, so I have to take minus that term. It'll be more mgr cos theta. And now what do I do? I want to apply, I repeat, d over dt of d of L over d of qi point or equal d of L over d of qi. So let's do it. I calculate d of L over d of theta point, I have mr square theta point.

Notes

Summary

3m 12s

Pendule mathématique



$$L = T - V = \frac{1}{2} m r^2 \dot{\theta}^2 + m g r \cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$ le moment cinétique

$$\frac{\partial L}{\partial \theta} = -m g r \sin\theta$$

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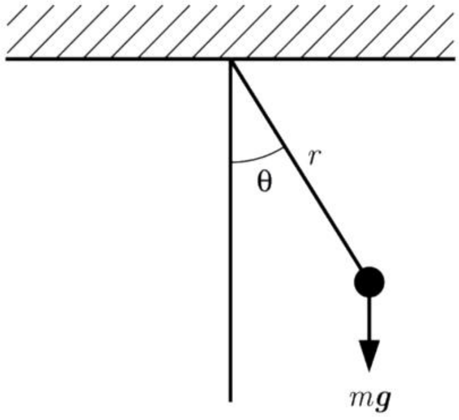
Maybe be careful here, we've often, very often calculated derivatives with respect to time, that's not what we're doing here. What this is, this symbol means, is to calculate the derivative of L with respect to the variable theta point. You consider theta point as a variable, and you derive that expression from L. So you only have mr square theta point. There's no question here of writing anything else. We need, oh, sorry, we recognize in this term the angular momentum, so we see, we'll, we'll point out here what's going to happen, right? When I calculate d of L on d of theta, I have to rivet, derive that with respect to theta. That's less mgr sin theta. Well, and the final formula tells me that the derivative of that with respect to time, is the derivative of angular momentum, my data is minus mgr sin theta. So we would like this term to be the projection on the normal of the gravitational moment. So indeed, we have a vector r, if we were doing the usual Newtonian mechanics, we would have defined the vector r like this. We will have a moment r cross F which tends to decrease the angle theta, and that's why there is the minus sign which appears here.

Notes

Summary

5m 27s

Pendule mathématique



$$L = T - V = \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \text{le moment cinétique}$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta \qquad \text{le moment de force}$$

$$\ddot{\theta} = -\frac{g}{r}\sin\theta$$

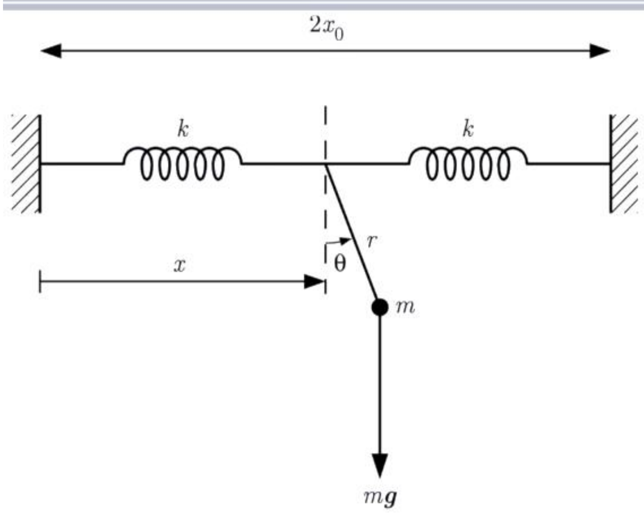
So once again, for Lagrange, you have to calculate the derivative of that with respect to time, and it must be equal to that. Sorry, I have the moment of force. And so, when I say that the derivative of that with respect to time, is equal to that, I simply, by dividing by mr squared, the theta point which is worth minus minus g on r sin theta, that's the formula we had. So, this is the result we would have obtained if we had applied the angular momentum theorem. What we have to recognize in the power of the method, is that for a given physical system, we only have to calculate the kinetic energy, the potential energy, and then we only have to make derivatives. And we don't need to decide anymore if we're going to apply the angular momentum theorem or the theorem, the, the, the, Newton's second law, since here we have a material point. Everything is done automatically by the choice of the generalized coordinates we have given ourselves.

Notes

Summary

7m 07s

Exemple à 2 degrés de liberté



$$V = \frac{1}{2}k(x - x_0)^2 + \frac{1}{2}k(x - x_0)^2 - mgr\cos\theta$$

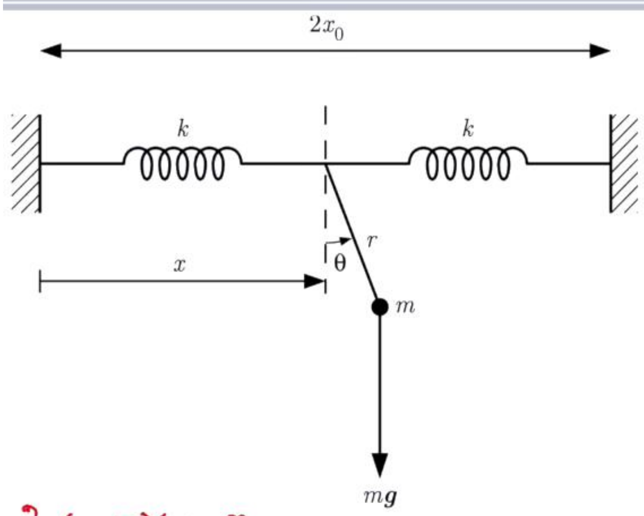
I propose you now to look at a problem with two degrees of freedom. Here is the physical system. There is a material point here, in the gravity field. It's a pendulum, and this point of attachment, we suppose that it slides on a horizontal line, and it's held by two springs. We suppose that x zero is the length at rest of the spring, that the spacing between these two walls is twice x zero. You'll see, it's just to make the writings cleaner that I took, I made this hypothesis which only simplifies the writings. How many degrees of freedom? Well here, we have two degrees of freedom, because it's not enough to give the theta inclination of the pendulum. You still have to know where this point of attachment is. And this point of attachment is not imposed, it will follow a certain dynamic which is imposed by the springs. So we need two degrees of freedom, we have to choose x and r. If now I calculate the potential energy, then, if I take the spring, this is the length x. For the spring on the left, it's simple, we have half of k times the elongation, it's x minus x zero squared, that's what I wrote here.

Notes

Summary

8m 12s

Exemple à 2 degrés de liberté



$2x_0 - x - x_0$

$x_0 - x$

$$V = \frac{1}{2}k(x-x_0)^2 + \frac{1}{2}k(x-x_0)^2 - mgr\cos\theta$$

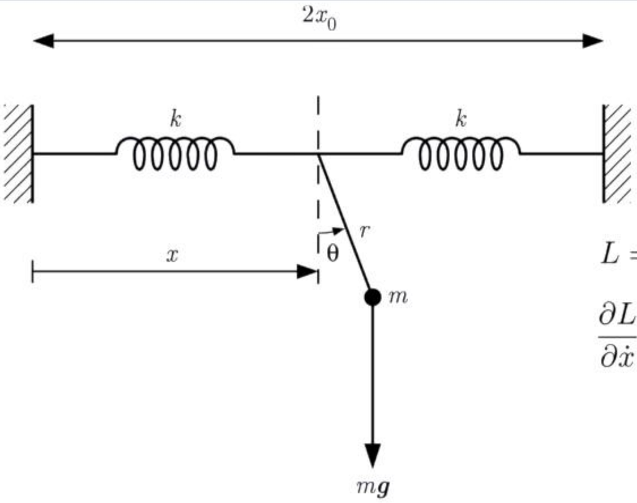
For the second spring, I have an elongation which is two x zero, this distance, minus x that gives me this length, and to this, you have to subtract the length at rest of the spring. So I subtract even less x zero. So I end up with x zero minus x, and when you square it you can write x minus x zero squared, that's the second term. For gravity, we have the same thing for the mathematical pendulum, we have a negative mgr cos theta, the potential is minus mgr cos theta, I removed the constant term that we usually put in the formula.

Notes

Summary

9m 50s

Exemple à 2 degrés de liberté



$$V = \frac{1}{2}\,k(x-x_0)^2 + \frac{1}{2}\,k(x-x_0)^2 - mgr\cos\theta$$

$$\boldsymbol{r} = \begin{pmatrix} x + r\sin\theta \\ -r\cos\theta \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} \dot{x} + r\dot{\theta}\cos\theta \\ +r\dot{\theta}\sin\theta \end{pmatrix}$$

$$L = \frac{1}{2}\,m\dot{x}^2 + \frac{1}{2}\,mr^2\dot{\theta}^2 + mr\dot{x}\dot{\theta}\cos\theta + mgr\cos\theta - k(x-x_0)^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\big(\dot{x} + r\dot{\theta}\cos\theta\big)$$

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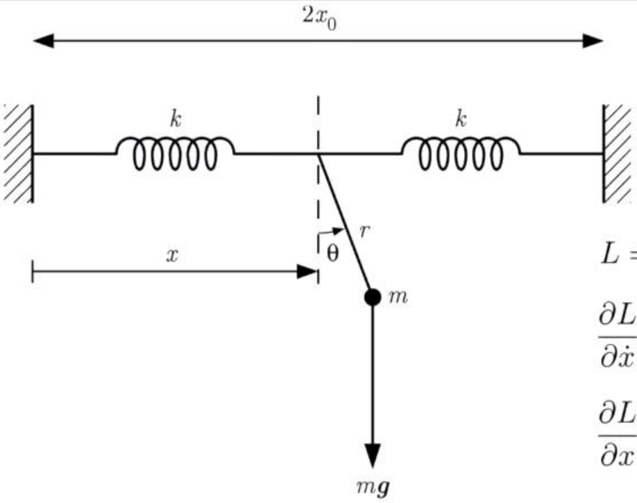
For the kinetic energy, it's a bit tricky, it's souv, always the kinematics that does, causes the most work, so I'm just going to write that the position vector it's x plus that distance, r sin theta, in the vertical direction, I'm going write a minus r cos theta, I derive with respect to time to get the velocity, it's just this, and, now, I'm ready to calculate the kinetic energy, and outright I can write the Lagrangian, I have this term squared, x point squared, there it is, I have here a half of m r square theta point squared cos square theta, but there I have the same thing in sin square theta, cos square theta plus sin square theta equals one, there is left, now I have the double product, which gives these terms, I have to calculate L, it was minus v, so I have to take minus this term, and minus these two terms which are the same, it makes minus k times x minus x zero squared. And now what do I do? So, let's remind it again. Lagrange is d over dt, from d of L over d of qi point, minus d of L over d of qi equals zero. So I have to now qi, it's one time it's x, and one time it's theta. So we have to do this derivation program twice. Let's go. I'm calculating d from L on d from x point.

Notes

Summary

10m 58s

Exemple à 2 degrés de liberté



$$V = \frac{1}{2}k(x-x_0)^2 + \frac{1}{2}k(x-x_0)^2 - mgr\cos\theta$$

$$\boldsymbol{r} = \begin{pmatrix} x + r\sin\theta \\ -r\cos\theta \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} \dot{x} + r\dot{\theta}\cos\theta \\ +r\dot{\theta}\sin\theta \end{pmatrix}$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mr\dot{x}\dot{\theta}\cos\theta + mgr\cos\theta - k(x-x_0)^2$$

$$\frac{\partial L}{\partial \dot{x}} = m(\dot{x} + r\dot{\theta}\cos\theta) \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + mr\dot{x}\cos\theta$$

$$\frac{\partial L}{\partial x} = -2k(x-x_0) \qquad \frac{\partial L}{\partial \theta} = -m\dot{x}r\dot{\theta}\sin\theta - mgr\sin\theta$$

$$m\ddot{x} + mr\ddot{\theta}\cos\theta - mr\dot{\theta}^2\sin\theta + 2k(x-x_0) = 0$$

$$mr^2\ddot{\theta} + mr\ddot{x}\cos\theta + mgr\sin\theta = 0$$

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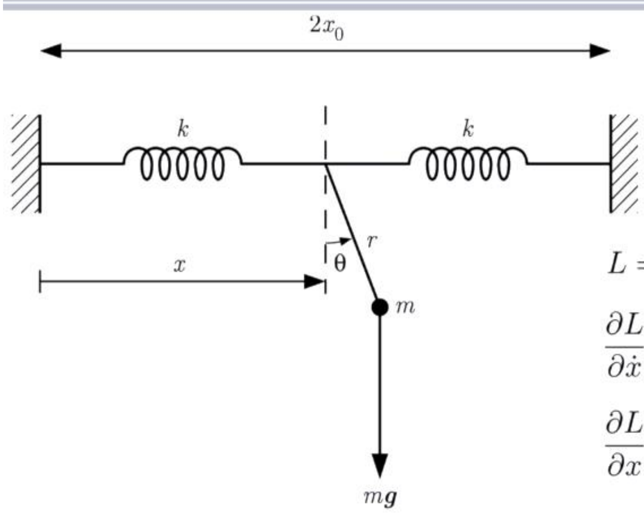
x point appears here, but it appears there too. So I have these two terms. I have to calculate d of L over d of x, which appears right there, the x. So now the derivative is quite simple. And now I have to do the derivative with respect to time of that term, minus that term equals zero. That gives me this. Indeed, I have an x dot dot. It's there. I have an r theta dot cos theta dot, it's there. I have minus r theta dot square sin theta, it's there. Minus this term, plus two k, x minus x zero equals zero. I have a two degree of freedom problem, so I have to do the derivatives now with respect to theta and theta point, d of L over d of theta point, there's a term here, but there's a second one from there. That's what I wrote there. The second term. I have to calculate d of L over d of theta, theta comes in here and there, there are two terms, they're there. And now I have to derive this with respect to time. So, and I have to make minus this, equals zero. So there's more than that left, why? Because here I have a mr x dot dot cos theta, which I wrote, but there's a mr x dot theta dot sin theta, with a minus sign, that one, has to come, but it's minus dL over d theta, so it'll have a plus sign, those two cancel out, and that's all that's left is that term, it's there.

Notes

Summary

13m 00s

Exemple à 2 degrés de liberté



$$V = \frac{1}{2}\,k(x - x_0)^2 + \frac{1}{2}\,k(x - x_0)^2 - mgr\cos\theta$$

$$\boldsymbol{r} = \begin{pmatrix} x + r\sin\theta \\ -r\cos\theta \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} \dot{x} + r\dot{\theta}\cos\theta \\ +r\dot{\theta}\sin\theta \end{pmatrix}$$

$$L = \frac{1}{2}\,m\dot{x}^2 + \frac{1}{2}\,mr^2\dot{\theta}^2 + mr\dot{x}\dot{\theta}\cos\theta + mgr\cos\theta - k(x - x_0)^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\big(\dot{x} + r\dot{\theta}\cos\theta\big) \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + mr\dot{x}\cos\theta$$

$$\frac{\partial L}{\partial x} = -2k(x - x_0) \qquad \frac{\partial L}{\partial \theta} = -m\dot{x}r\dot{\theta}\sin\theta - mgr\sin\theta$$

$$m\ddot{x} + mr\ddot{\theta}\cos\theta - mr\dot{\theta}^2\sin\theta + 2k(x - x_0) = 0$$

$$mr^2\ddot{\theta} + mr\ddot{x}\cos\theta + mgr\sin\theta = 0$$

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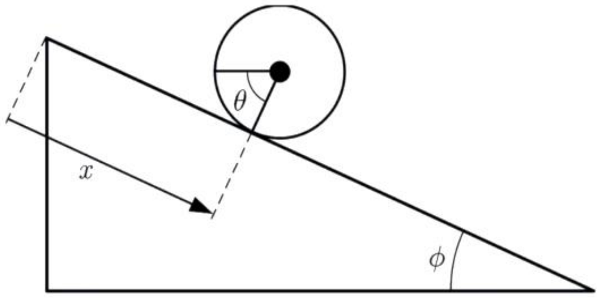
And I'm done with this problem, I've written the equations of motion for this problem with two degrees of freedom.

Notes

Summary

14m 51s

Cylindre roulant sans glisser



$$L = \frac{1}{2} I_\Delta \dot{\theta}^2 + \frac{1}{2} M\dot{x}^2 + Mgx\sin\phi$$

$L(q, \dot{q}, \cancel{x})$

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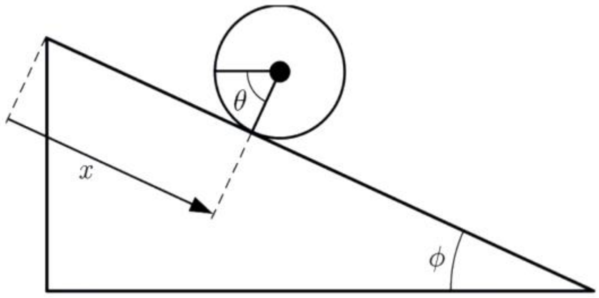
I finish with a problem of an undeformable solid, I suppose to examine the problem of a cylinder, which goes down an inclined plane. So, a good practice of problem solving with Lagrange, is to write the Lagrangian first with the variables that suit us best. And the simplest way is to write for the kinetic energy half of m, x point square, x is the coordinate of the center of mass, so there we have the kinetic energy that we would have if all the mass was at the center of mass, and I suppose that my cylinder has a moment of inertia i delta, I use the theta angular variable to express the kinetic energy of the rotation of the system around its center of mass. For the energy, for the, for the gravity, potential of gravity, I have to express mg times the height, so if I write mgx, sin phi, so I have a term like this, which is downwards, so the v would be negative, but L it was minus v, so it becomes more mgx sin phi. And now, before I get into the derivations, I have to clean up my formula so that I really have an L that is a function of q, of q point, possibly of time, which is not the case here, but I only want to have q and q point, I don't want to have auxiliary variables.

Notes

Summary

15m 01s

Cylindre roulant sans glisser



$$L = \frac{1}{2} I_\Delta \dot{\theta}^2 + \frac{1}{2} M\dot{x}^2 + Mgx\sin\phi$$

Roulement sans glissement : $x = R\theta$

1 degré de liberté

$$L = \frac{1}{2} M \left(1 + \frac{I_\Delta}{MR^2}\right) \dot{x}^2 + Mgx\sin\phi$$

$$\ddot{x} = \frac{g\sin\phi}{\left(1 + \frac{I_\Delta}{MR^2}\right)}$$

$$\frac{\partial L}{\partial \dot{x}} = M\left(1 + \frac{I_\Delta}{MR^2}\right)\dot{x} \qquad \frac{\partial L}{\partial x} = Mg\sin\phi$$

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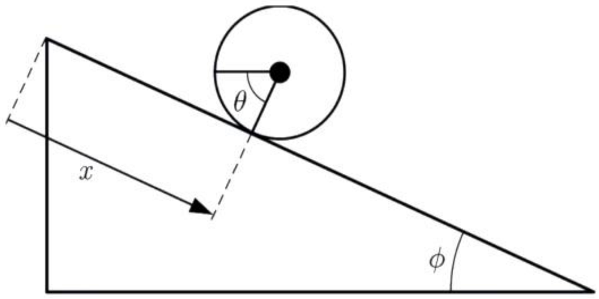
Now this is what I do, I use the condition of rolling without slip, which I can express as x being equal to R times theta, R, I could have written it down here, the radius of the cylinder, like this, and so I have x point which is worth R theta point, if I now choose x as the generalized coordinate, I have to get rid of the theta point here, because I have one degree of freedom. So if I use the theta point equals x point on R,, I have here, once I put my terms in order, I have a half of m, x point squared, with the one, it's if we had a material point and that's due to the fact that we have on solid. And there I have x. I apply Lagrange, I have to calculate d of L on d of x point, that gives me this term, and then I have to calculate d of L on d of x, there is no x, there is only one there, it's very simple. And then I write that the derivative with respect to time of this term must be equal to this term, and I simplify the expressions a little bit, we find this, we find that everything happens as if we had a fall, in a constant field which is no longer g but g sin phi because of the inclination of the tilted plane, and then there is still this normalization term underneath, I delta divided by an R square which appears.

Notes

Summary

16m 49s

Cylindre roulant sans glisser



$$L = \frac{1}{2} I_\Delta \dot{\theta}^2 + \frac{1}{2} M \dot{x}^2 + M g x \sin\phi$$

Roulement sans glissement : $x = R\theta$

1 degré de liberté

$$L = \frac{1}{2} M \left(1 + \frac{I_\Delta}{MR^2}\right) \dot{x}^2 + M g x \sin\phi$$

$$\ddot{x} = \frac{g \sin\phi}{\left(1 + \frac{I_\Delta}{MR^2}\right)}$$

$$\frac{\partial L}{\partial \dot{x}} = M \left(1 + \frac{I_\Delta}{MR^2}\right) \dot{x} \qquad \frac{\partial L}{\partial x} = M g \sin\phi$$

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It is indeed a term without unit. And here is an example of application for an undeformable solid, we have adopted the same algebraic approach, we have written L, and then we do derivatives.

Notes

Summary

18m 38s