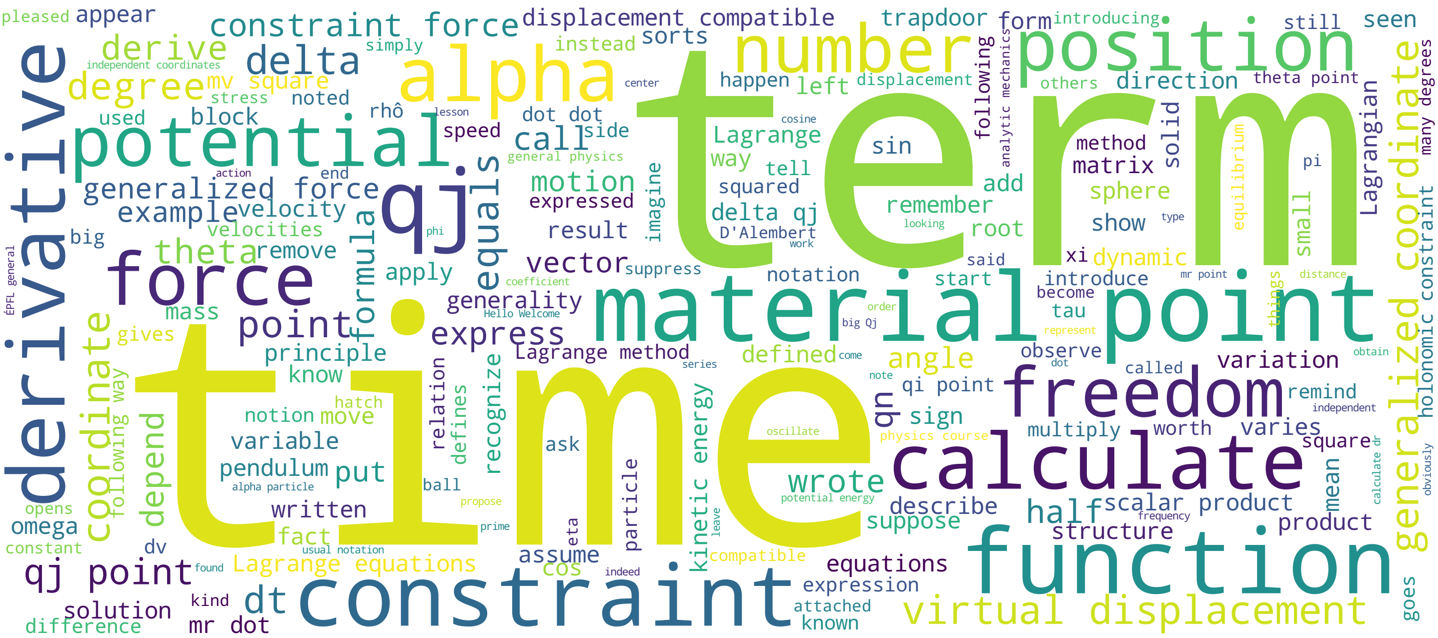
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Méthode de Lagrange

Mécanique, cours 25.1

Jean-Philippe Ansermet

http://go.epfl.ch/traite-meca-4-2



Video

EPFL

- Contraintes
- Nombre de degrés de liberté
- Déplacements virtuels compatibles
- Forces généralisées
- Equations de Lagrange

Hello. Welcome to the ÉPFL general physics course. In this lesson, I will introduce an extremely efficient method to obtain the equations of motion. I call it the Lagrange method, and the equations are known as the Lagrange equations. This chapter, typically, is part of a course called analytic mechanics, analytic mechanics allows to reveal all kinds of fundamental, important structures of mechanics, but here, what I introduce, basically only serves to give a method to solve the problems of mechanics you have seen so far. First, we will consider a mechanical system with simple constraints, we will see how many degrees of freedom the system has, we will have to define what we call a degree of freedom, we will see the virtual displacements compatible with the constraints of the problem, this will allow us to define what I'm going to call generalized forces, we'll have generalized coordinates, we'll have generalized forces, and finally we'll see Lagrange's equations, which we'll be able to apply to all sorts of situations.

Notes

Summary

0m 03s

Définition : contraintes

N points matériels

positions : $\boldsymbol{r}_\alpha$, $\alpha = 1, \ldots, N$

contraintes exprimables sous la forme d'un ensemble de k équations :

$$\begin{aligned} f_1(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \\ f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \\ &\vdots \\ f_k(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \end{aligned}$$

k : le nombre de contraintes

dites : "holonômes"

Ces contraintes peuvent dépendre explicitement du temps

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I start with the constraints. So, here is the problem I have. I have N material points. I locate the position of these big N material points with the vector r alpha. Alpha equals one to N, which is the notation we have used up to now for a system of material points. Now I imagine that my system is such that I have geometric constraints which I can express in the following way. You see here a series of equations, they are functions of the positions r one, r two, r N, of the N material points, this function can also be a function of time, and I have k, equation of this type. Big N is the number of material points, k is the number of constraints. When we have constraints which are expressed by functional relations of positions only, we say that we have holonomic constraints. We could have constraints on the velocities, that would be different, than what we have here. Here we have what is called in the literature holonomic constraints. Once again, these constraints can depend explicitly on time, we have to see a nuance here, we are going to suppose a certain dynamics to this system of N material points.

Notes

Summary

1m 27s

Définition : contraintes

N points matériels

positions : $\boldsymbol{r}_\alpha$, $\alpha = 1, \ldots, N$

contraintes exprimables sous la forme d'un ensemble de k équations :

$$\begin{aligned} f_1(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \\ f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \\ &\vdots \\ f_k(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \end{aligned}$$

k : le nombre de contraintes

dites : "holonômes"

Ces contraintes peuvent dépendre explicitement du temps

Mécanique | 2013 15

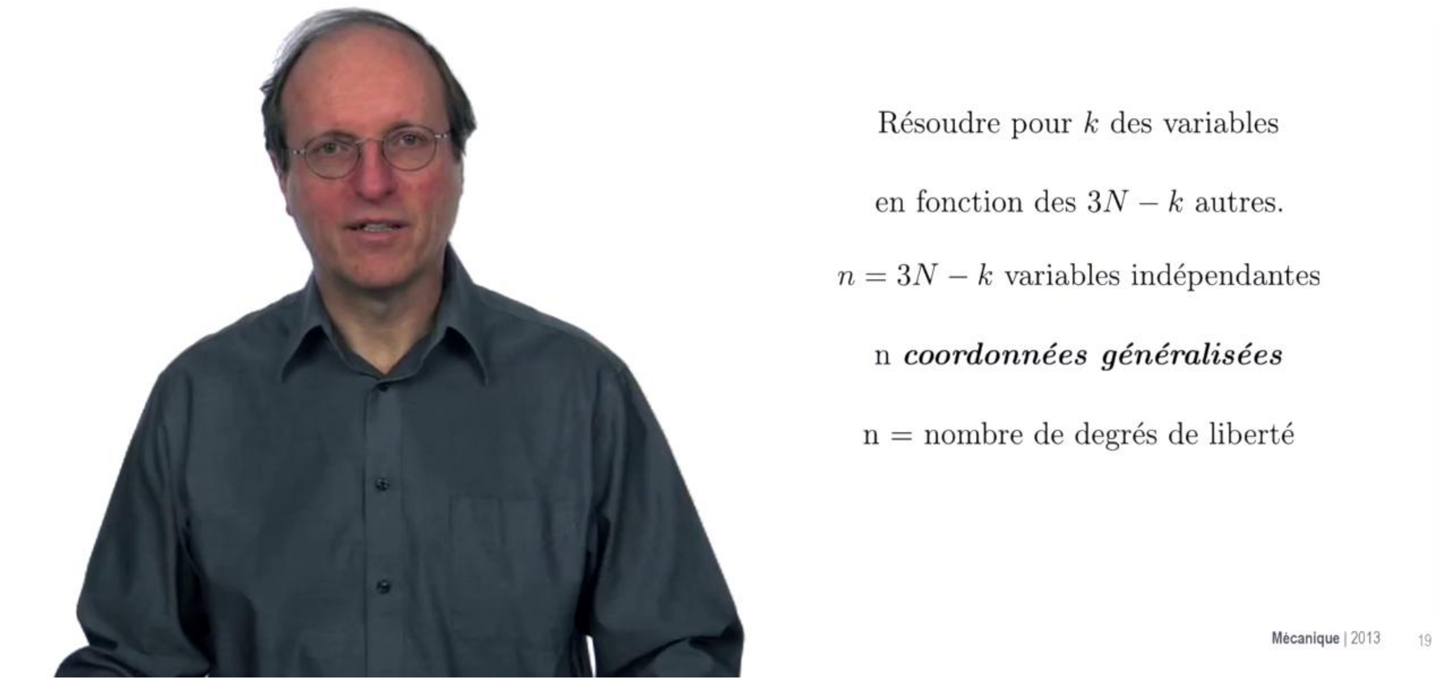
So r one, r two, r N will depend on time, this is what we call an implicit time dependence, but we can have an explicit time function, which is represented here by the fact that time becomes an explicit variable of this function. For example, we saw the problem of the ball which is constrained to move in a ring and the ring turns around a vertical axis, there is an example of a constraint which depends on time, explicitly.

Notes

Summary

3m 09s

Définition : nombre de degrés de liberté

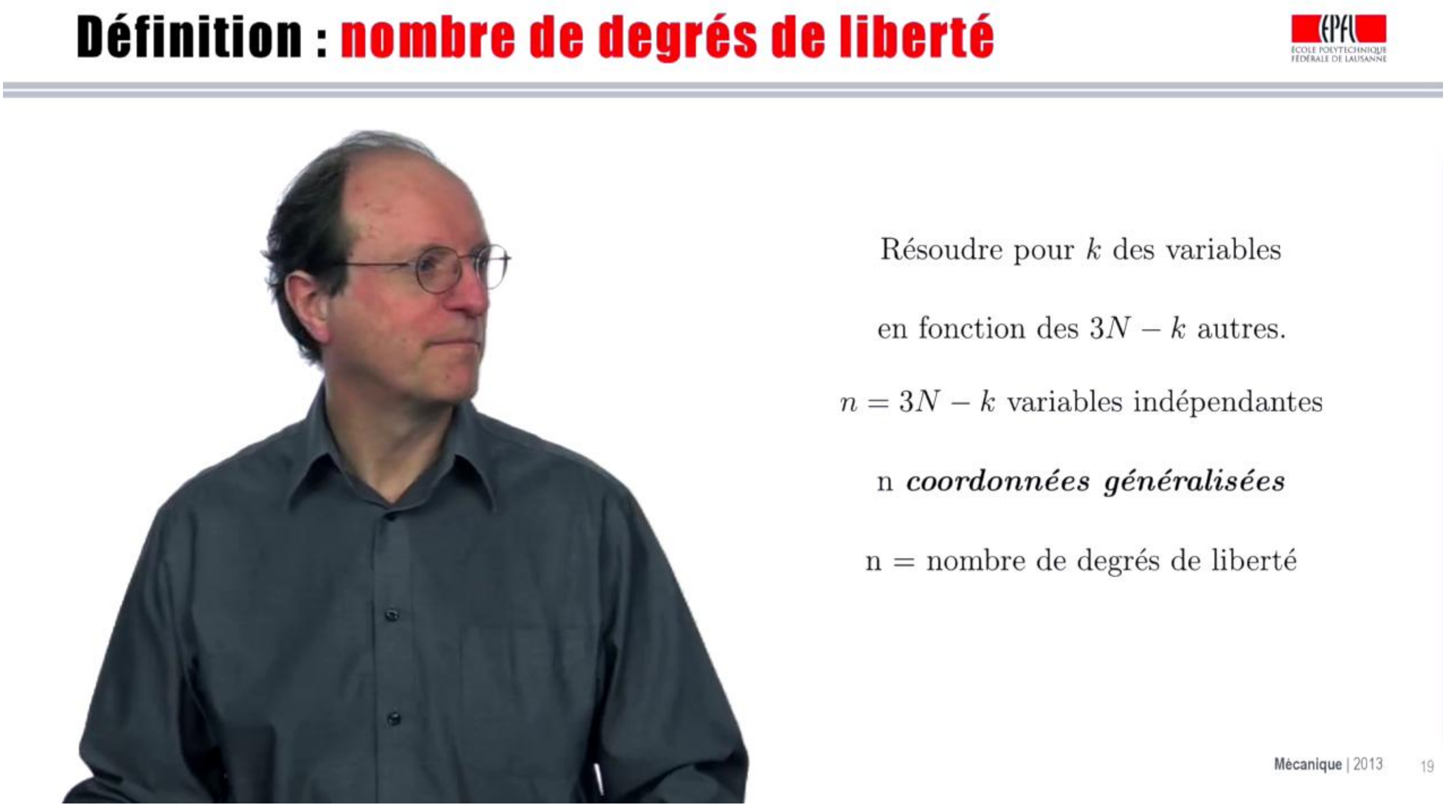


So now I define this extremely important notion in lagrangian mechanics, the notion of degrees of freedom. This notion can be found in all sorts of fields of physics. Here, we gave ourselves k constraints. And we had N material points, so we had three N coordinates to represent the positions of these N material points, we can choose to express k of these three N coordinates in function of the others. At this point we are left with three N minus k independent coordinates. This number three N minus k of independent coordinates that we need, and that are sufficient to describe the position of the system of material points, this number that I have noted as small n, is what we call the number of degrees of freedom. We will use generalized coordinates, n, small n generalized coordinates, to define the position of the material points of the system. Let's take an example. You have a material point, only one, forced to move on a sphere. You need two angles to define the position of this material point so we will have two degrees of freedom. If now this material point is attached to a wire, of fixed length, without mass, which is attached to the North pole, of this sphere, then you have only one degree of freedom.

Notes

Summary

3m 42s



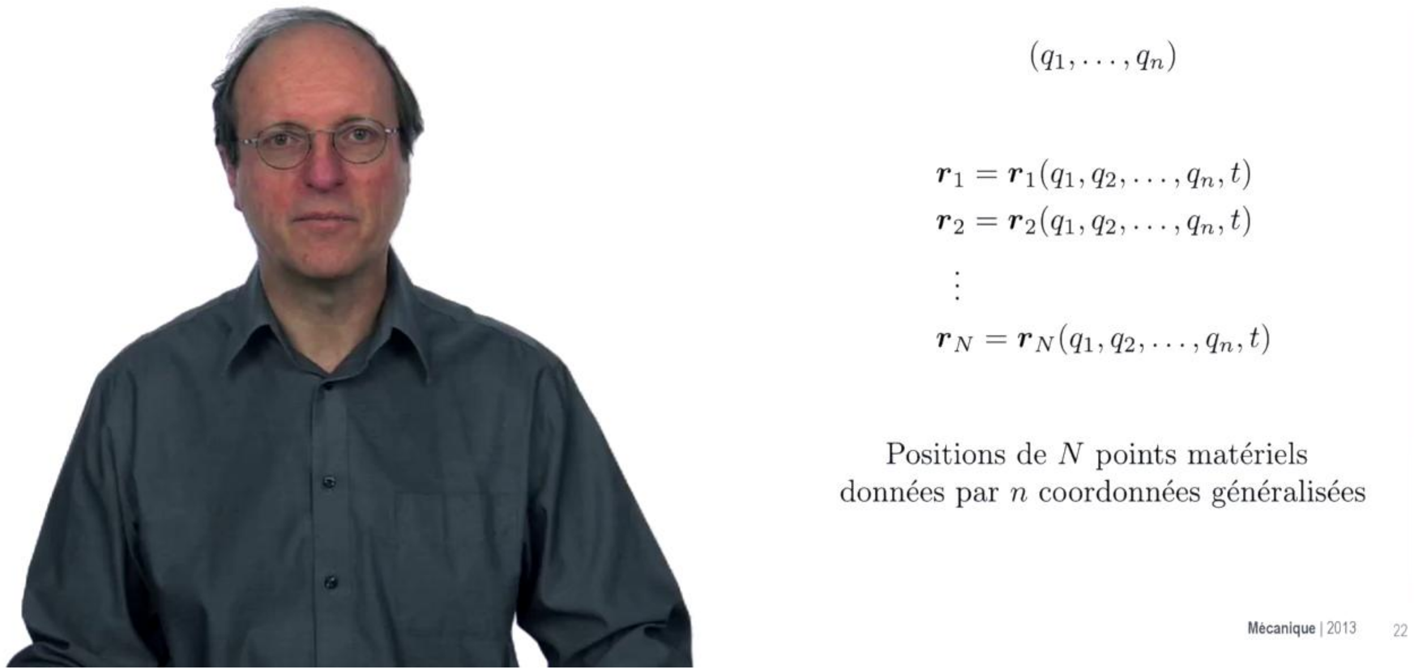
Your material point is still moving on a two-dimensional surface, the sphere, but only one angle is needed, only one generalized coordinate to describe the position of the material point. So in this case, the second case, we will say that n is one.

Notes

Summary

5m 41s

Définition : coordonnées généralisées



So, now the usual notation is to write down these coordinates, why generalized? Because it can be either a distance, or an angle. The usual notation is to note them q one to qn. And now the positions of our big N material points, are given as functions of the generalized coordinates, and of the time, the time being able to appear explicitly in these functions. To remember this notation a little better, if you think about a solid, the number of degrees of freedom of a solid, as we have seen, is between one and six. The number of material points that define this solid can be much, much larger. So I used big N for the number of material points, small n for the number of degrees of freedom. Let's take an example of time-dependent constraints.

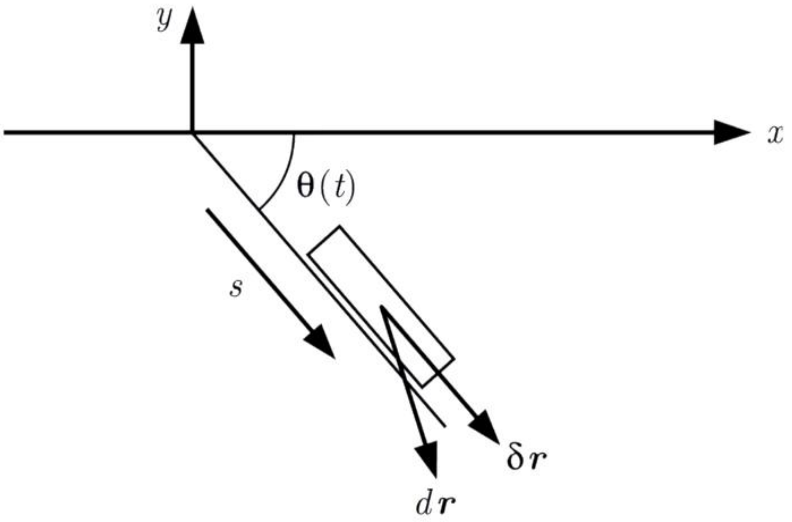
Notes

Summary

6m 03s

Exemple de contrainte dépendant du temps

déplacements virtuels compatibles avec les contraintes



$\delta \boldsymbol{r}(t)$

I invite you to consider the following situation. We imagine, that this is a horizontal, and that there is a trap door that opens in this floor. Here you have the trapdoor. And here I've defined the angle and I assume here that I have something like times t, with a a constant, so I have a time equation that's set. What is the number of degrees of freedom in this problem? So, I apologize. Here, instead of a material point I have a block. This is the object so I want to study the dynamics. How many degrees of freedom do I have for this block? Well, there is one. We move in a vertical plane, that's a two dimensional space, but the theta coordinate being fixed, there is only this displacement marked by the s coordinate, which is our generalized coordinate. And now I ask the following question: what is the displacement I would have, I'm talking about a virtual displacement, what is the displacement I would have, if the block moved along the constraint when I remove the motion from the constraint? Well I've written it down here delta r. it's a slip along this trapdoor. This, I call the virtual displacement compatible with the stress.

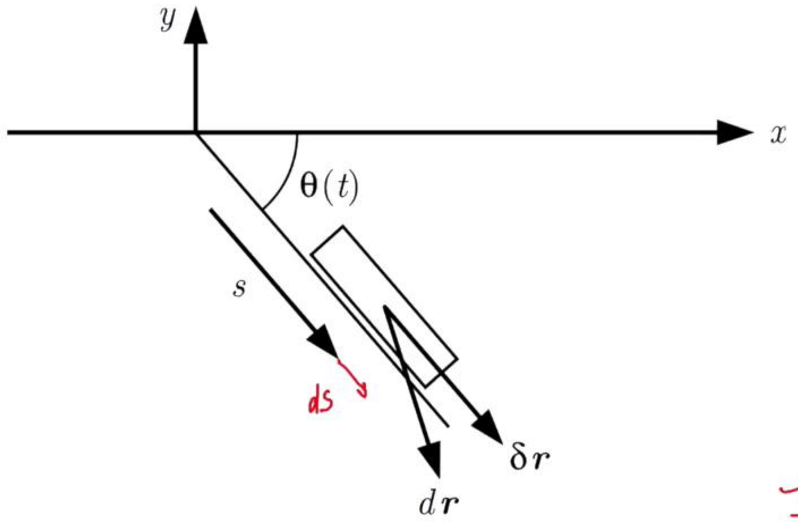
Notes

Summary

7m 02s

Exemple de contrainte dépendant du temps

déplacements virtuels compatibles avec les contraintes



$$\delta \boldsymbol{r}(t)$$

$$\neq \boldsymbol{v} dt$$

si les contraintes dépendent du temps

$$x = s\cos(at) \qquad \frac{\partial x}{\partial s} = \cos(at)$$

$$y = -s\sin(at) \qquad \frac{\partial y}{\partial s} = -\sin(at)$$

$\vec{r}$

$$\delta \vec{r} = \frac{\partial \vec{r}}{\partial s} ds$$

When you have a constraint like this hatch, that opens, this virtual displacement delta r, is not the same thing as vdt, that would be dr, marked here. Why? Because in a time dt, there is indeed a displacement of the block along the hatch, but there is also the fact that the hatch goes down. So we have a d of r that is distinct from the delta r. To make things more concrete, let's look at the position coordinates x and y, x and y being the coordinates here of, of the center of mass of this block. So I will write x equals s cos at, where at is the value of the theta angle, as a function of time, and y equals minus sin at. If I now calculate the derivative of x with respect to s, you have cos at. If I calculate the derivative of y with respect to s, I have minus sin at. And you see that here we have cos at minus sin at, it's a unit vector in that direction, so in the direction of displacement. So when I take a vector r like this, and now I calculate d from r. Here, I've calculated d of r over d of s. I have a vector in the direction tangent to the stress. And so, if I multiply by ds, if I have a displacement ds like this, then I get my delta r. So, we can see how we calculate a delta r in this case, which is a virtual displacement compatible with the constraints.

Notes

Summary

8m 49s

Définition : déplacements compatibles

$$\delta \boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(q_1 + \delta q_1, \ldots, q_n + \delta q_n, t) - \boldsymbol{r}_\alpha(q_1, \ldots, q_n, t)$$

$$\delta \boldsymbol{r}_\alpha = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}\, \delta q_j$$

How can we define the virtual displacement compatible with the constraints in all generality? Well, we will consider the position vector of the alpha particle when the variables, the generalized coordinates q un, qn are increased by a certain delta q un, delta qn. And we make the difference with the position of the alpha particle when we are at a position given by q un and qn. Again, if we only had one coordinate that varies, like the s-coordinate for the trapdoor problem. We just have here, the, the, s that varies by some delta s, and we compare it with the r that we have when delta s equals zero. And that gives us the compatible virtual displacement. But here, we have several, we suppose in all generality, that we have a displacement which is defined by several generalized coordinates. And so, if these displacements are infinitesimal, I can write this delta r in the following way : I calculate the derivative d of r on d of qj, and I could have written j equal to one to n, the number of degrees of freedom. And I multiply by delta qj. Basically, to calculate this difference, I look at the variation when q one varies, the variation when, variation of the position when q one varies, when q two varies, when qn varies. And I express it with this rule of limited development that we know well now.

Notes

Summary

11m 11s

Principe de d'Alembert

décomposition $\boldsymbol{F}^{tot} = \boldsymbol{F}^{cont} + \boldsymbol{F}$

d'abord : un seul point matériel

$$\boldsymbol{F} + \boldsymbol{F}^{cont} - m\frac{d\boldsymbol{v}}{dt} = 0$$

$\delta\boldsymbol{r}$ virtuel compatible avec les contraintes

$$\left(\boldsymbol{F} + \boldsymbol{F}^{cont}\right) \cdot \delta\boldsymbol{r} - m\frac{d\boldsymbol{v}}{dt} \cdot \delta\boldsymbol{r} = 0$$

hypothèse $\boldsymbol{F}^{cont} \cdot \delta\boldsymbol{r} = 0$

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Now, I'm going to try to describe the dynamics where I suppress the constraint forces. We will see at the end of the theory, that we can very well calculate the constraints by the Lagrange method, but in a first time, we will try to suppress them. So, I'm going to make a decomposition of the forces. I'm going to assume that I have constraint forces and other forces. I use the symbol, simply F for all the other forces, and so the total forces. That's the stress forces, plus all the other F forces. Now, I write for a single material point F equals ma, and here I wrote force minus mass times acceleration equals zero. I consider a virtual displacement compatible with the constraints, and I multiply Newton's equation by this multiplication, so it is scalar product by this virtual displacement vector compatible with the constraints, like this. And in that equation, I have a scalar product of the constraint forces times the virtual displacements. So, you remember we said that the constraint forces are forces normal to the curve or surface that defines the constraint. Otherwise, it would be another physics that this force would describe. So, we have this scalar product of the constraint forces of the virtual displacements which is zero.

Notes

Summary

13m 12s

Principe de d'Alembert

décomposition $\quad \boldsymbol{F}^{tot} = \boldsymbol{F}^{cont} + \boldsymbol{F}$

d'abord : un seul point matériel

$$\boldsymbol{F} + \boldsymbol{F}^{cont} - m\frac{d\boldsymbol{v}}{dt} = 0$$

$\delta\boldsymbol{r}$ virtuel compatible avec les contraintes

$$\left(\boldsymbol{F} + \boldsymbol{F}^{cont}\right) \cdot \delta\boldsymbol{r} - m\frac{d\boldsymbol{v}}{dt} \cdot \delta\boldsymbol{r} = 0$$

hypothèse $\quad \boldsymbol{F}^{cont} \cdot \delta\boldsymbol{r} = 0$

$$\left(\boldsymbol{F} - m\frac{d\boldsymbol{v}}{dt}\right)\delta\boldsymbol{r} = 0$$

pour un système de points matériels

$$\sum_{\alpha}\left(\boldsymbol{F}_{\alpha} - m_{\alpha}\frac{d\boldsymbol{v}_{\alpha}}{dt}\right) \cdot \delta\boldsymbol{r}_{\alpha} = 0$$

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Then we have this: this is for a material point, and if we have several material points, we just have to sum over all the material points. We have an expression of the dynamics which is more complex than before, but we have somehow simplified the problem because we have removed all the constraint forces.

Notes

Summary

14m 53s

Définitions : forces généralisées

$$\sum_{\alpha}\left(\boldsymbol{F}_{\alpha}-m_{\alpha}\frac{d\boldsymbol{v}_{\alpha}}{dt}\right)\cdot\delta\boldsymbol{r}_{\alpha}=0$$

$$\boldsymbol{F}_{\alpha}\cdot\delta\boldsymbol{r}_{\alpha}$$

$$\sum_{j=1}^{n}\sum_{\alpha=1}^{N}\boldsymbol{F}_{\alpha}\cdot\frac{\partial\boldsymbol{r}_{\alpha}}{\partial q_j}\,\delta q_j\quad=\sum_{j=1}^{n}Q_j\,\delta q_j$$

$$Q_j=\sum_{\alpha=1}^{N}\boldsymbol{F}_{\alpha}\cdot\frac{\partial\boldsymbol{r}_{\alpha}}{\partial q_j}\quad(j=1\cdots n)$$

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Now, I'm going to look at the principle, this equation is known as D'Alembert's principle. I will first look at this term of forces, scalar product with compatible displacements. This is the other forces of the system, not the constraint forces. I want to look at these terms, and what do I do? I'm going to express, I remind you that we said delta r alpha, the virtual displacement compatible with the constraints, it was the sum of all the generalized coordinates of the derivatives, of the positions derived with respect to the qj coordinate times a small displacement, a small change delta qj of this coordinate. This formula, this is the one I put here, and I put the sum on j in front. And now, I observe that I can write the following. I can write sum over j of that times delta qj. I wrote big Qj times delta qj, where this big Qj, this is what we call the generalized force, this, this sum over the alpha material points equals one to big N of these products there.

Notes

Summary

15m 17s

$$\sum_{\alpha}\left(\boldsymbol{F}_{\alpha}-m_{\alpha}\frac{d\boldsymbol{v}_{\alpha}}{dt}\right)\cdot\delta\boldsymbol{r}_{\alpha}=0$$

$$m\frac{d\boldsymbol{v}}{dt}\cdot\delta\boldsymbol{r}\quad=\sum_{j}\left[\frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right)-\frac{\partial}{\partial q_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right]\delta q_j\qquad\text{A démontrer !}$$

$$m\ddot{\boldsymbol{r}}\cdot\delta\boldsymbol{r}=\sum_{j}m\ddot{\boldsymbol{r}}\cdot\frac{\partial\boldsymbol{r}}{\partial q_j}\delta q_j\quad=\sum_{j}\left(\frac{d}{dt}\left(m\dot{\boldsymbol{r}}\cdot\frac{\partial\boldsymbol{r}}{\partial q_j}\right)-m\dot{\boldsymbol{r}}\cdot\frac{d}{dt}\left(\frac{\partial\boldsymbol{r}}{\partial q_j}\right)\right)\delta q_j$$

Now, I take again my principle of D'Alembert. And now, I'm going to focus my attention on the kinetic terms, so the terms with the speed. So there, I'm a bit sorry, I like this chapter a lot. You'll like it a lot; I know that from having taught for many years, but the passage that comes now is a bit heavy. So I propose, first to give the result, and then to demonstrate it. So, what I'm going to show is this. I'm going to show for a mass, so we're going to remove the sums on alpha, and it's quite complicated, size sense. I'm going to show that this term here, I can write it like this. So you recognize something nice in this formula, it's the half of mv square, the kinetic energy of the particle of mass m. You see that you have a sum over j, sum over the, over displacements of the generalized coordinate variations. Here I have rewritten m dv over dt as mr point point. And now, the delta r; so I engage in the demonstration of this formula. The delta r, I apply its definition, so I have a sum over j coming. Then, I say to myself that this term, I can write it as the derivative with respect to time of mr point times that, that's what I wrote here. But when I write that, I'm introducing the term mr dot dot that we have here.

Notes

Summary

16m 49s

$$\sum_\alpha \left(\boldsymbol{F}_\alpha - m_\alpha \frac{d\boldsymbol{v}_\alpha}{dt}\right)\cdot \delta \boldsymbol{r}_\alpha = 0$$

$$m\frac{d\boldsymbol{v}}{dt}\cdot\delta\boldsymbol{r} \quad = \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right]\delta q_j \qquad \text{A démontrer !}$$

$$m\ddot{\boldsymbol{r}}\cdot\delta\boldsymbol{r} = \sum_j m\ddot{\boldsymbol{r}}\cdot\frac{\partial \boldsymbol{r}}{\partial q_j}\,\delta q_j \quad = \sum_j \left(\frac{d}{dt}\left(m\dot{\boldsymbol{r}}\cdot\frac{\partial \boldsymbol{r}}{\partial q_j}\right) - m\dot{\boldsymbol{r}}\cdot\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}}{\partial q_j}\right)\right)\delta q_j$$

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_n, t) \qquad \boldsymbol{v} = \sum_j \frac{\partial \boldsymbol{r}}{\partial q_j}\,\dot{q}_j + \frac{\partial \boldsymbol{r}}{\partial t} \qquad \Longrightarrow \quad \frac{\partial \boldsymbol{v}}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}}{\partial q_j}$$

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But I also introduce the term, because we have a product here, so we're going to have two terms. And when I write this term, I'm introducing a mr dot times the d on dt of that that we didn't have, so I have to remove it. That's what I wrote here. Here, we have mr dot, d on dt of that, and I take it out. Well, now we have to work on these terms and this one to get to the final result. So first, I notice that if r; we had seen this structure of r expressed as a function of generalized coordinates and time. If r is like this, then the velocity, to calculate the velocity, as q one is a function of time up to qn function of time too. To calculate dr over dt, you have to calculate dr over d from qj times qj point, and then for the last term, d from r over dt. Be careful that there, we have a round d, that is to say that we make the derivative where t appears explicitly in this function. Now in this formula, we remember that r as it is written here is a function of q but not of q point. So, looking at this formula, we'll say that if I have to calculate d of v over d of qj point, sorry, d of v over d of qj point, which I've written here, well, neither this nor that depends on the qj points so all that's left is d of r over d of qj.

Notes

Summary

18m 46s

$$\sum_\alpha \left(\boldsymbol{F}_\alpha - m_\alpha \frac{d\boldsymbol{v}_\alpha}{dt} \right) \cdot \delta \boldsymbol{r}_\alpha = 0$$

$$m\frac{d\boldsymbol{v}}{dt} \cdot \delta \boldsymbol{r} \quad = \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} m \boldsymbol{v}^2 \right) \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} m \boldsymbol{v}^2 \right) \right] \delta q_j \qquad \text{A démontrer !}$$

$$m\ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} = \sum_j m\ddot{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} \delta q_j \quad = \sum_j \left(\frac{d}{dt}\left(m\dot{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} \right) - m\dot{\boldsymbol{r}} \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}}{\partial q_j} \right) \right) \delta q_j$$

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_n, t) \quad \boldsymbol{v} = \sum_j \frac{\partial \boldsymbol{r}}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}}{\partial t} \qquad \Longrightarrow \; \frac{\partial \boldsymbol{v}}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}(q_1 \ldots q_i \ldots q_n, t)}{\partial q_j} \right) = \sum_i \left\{ \frac{\partial^2 \boldsymbol{r}}{\partial q_i \partial q_j} \dot{q}_i \right\} + \frac{\partial^2 \boldsymbol{r}}{\partial q_j \partial t} = \frac{\partial}{\partial q_j} \left\{ \sum_i \frac{\partial \boldsymbol{r}}{\partial q_i} \dot{q}_i + \frac{\partial \boldsymbol{r}}{\partial t} \right\} = \frac{\partial \boldsymbol{v}}{\partial q_j}$$

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We have a strange formula here that we're not used to, which comes from the structure we've given ourselves here. So, the d of r on d of qj that I have here, I'll be able to replace by d of v on d of qj point and you will see that this, it brings us closer to this result. Now, we still need to see what to do with this. So, the d over dt of the d of r over d of qj where I explicitly put the variables here. I need to calculate, now, the derivative with respect to q un, qi, qn. I wrote here sum over i d square r over dqi dqj and I sum over i times the qi point. And then, I also have the d of r over d of qj which is a function explicitly of time. Besides I have to put this term, it's there. Now I have a d of qj here and I have a d of qj there. I put forward this derivative with respect to qj and I'm left with d of r over d of qj, sorry, d of r over d of qi, qi point and there I have a d of r over d of t, that term there. And there, I recognize the speed. So I have d of v over d of qj. So here's a second result that's a little weird, d over dt by d by r over d by qj, that's worth d by v over d by qj. That I'll use here. I'm rewriting the partial results.

Notes

Summary

20m 39s

$$m\ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} = \sum_j m\ddot{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j}\,\delta q_j \quad = \sum_j \left(\frac{d}{dt}\left(m\dot{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} \right) - m\dot{\boldsymbol{r}} \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}}{\partial q_j} \right) \right) \delta q_j$$

$$m\dot{\boldsymbol{v}} \cdot \delta \boldsymbol{r} = \sum_j \left(\frac{d}{dt}\left(m\boldsymbol{v} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} \right) - m\boldsymbol{v} \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}}{\partial q_j} \right) \right) \delta q_j$$

$$m\dot{\boldsymbol{v}} \cdot \delta \boldsymbol{r} = \sum_j \left[\frac{d}{dt}\left(m\boldsymbol{v} \frac{\partial \boldsymbol{v}}{\partial \dot{q}_j} \right) - m\boldsymbol{v} \frac{\partial \boldsymbol{v}}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m\boldsymbol{v}^2 \right) \right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m\boldsymbol{v}^2 \right) \right] \delta q_j \qquad \text{cqfd}$$

This is the term I care about. Here I simply applied the relation that defines the virtual displacement compatible with the constraints. I arrived at this result. I rewrite it here where I write instead of r dot dot, I write v dot. Here I recognize a v and here I recognize a v too. And now, this d of r on d of qj, I replace it with d of v on d of qj dot, and this term there, I replace it with d of v on d of qj. And now, if I concentrate on that term, I have to recognize the derivative with respect to qj point of one half of mv square. That's what I wrote here. If you have to do this derivative with respect to qj point, you're going to have one half times the two times m times v times the derivative of v with respect to qj point. That's what's there. In an analogous way there, we can say that this term here is d on d of qj by one half of mv square. Well, I am pleased to tell you that we have succeeded in proving the formula I had announced. As I said at the beginning, it's a little bit tricky. So now we are ready.

Notes

Summary

22m 35s

Equations de Lagrange (1)

Pour un point matériel :

$$m\dot{\boldsymbol{v}} \cdot \delta\boldsymbol{r} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right]\delta q_j$$

$$\boldsymbol{F} \cdot \delta\boldsymbol{r} = \sum_{j=1}^{n} Q_j\,\delta q_j$$

$$\sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0$$

Pour un système de points matériels :

$$T = \sum_{\alpha=1}^{N}\frac{1}{2}m_\alpha \boldsymbol{v}_\alpha^2$$

$$Q_j = \sum_{\alpha=1}^{N}\boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}$$

q_j indépendants $\Longrightarrow$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

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We started from D'Alembert, we have this term that we can express in this way, we have the term of force that we can express with generalized forces and if we simply write that half of mv square is the kinetic energy T and the equation of D'Alembert says that minus that equals zero so I have this term minus the term of force, sum over j times the delta qj equal to zero. And now, these generalized coordinates are independent of each other. I must have this equation whatever choice I make for the delta qj. So what I need is for the term in the parenthesis to be zero. And I'm pleased to tell you that this is the first form of Lagrange's equations. If now you have several particles, you have to add in all that I have done sums over alpha. You'll bring up the kinetic energy, the sum of one half of m alpha, v alpha squared, sum over all the particles. And, as we have already seen, for the generalized force you just have to sum these terms over all the alphas. And, the formula comes back to the same thing and you have the same conclusion: the Lagrange equations of the first type.

Notes

Summary

24m 09s

Equations de Lagrange (2)

Forces conservatives :

$V(x_{\alpha 1}(q_1, \dots q_n), x_{\alpha 2}(q_1 \dots q_n) \dots\dots)$

$$Q_j = \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \qquad Q_j^{pot} = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial V_\alpha}{\partial x_\alpha^i}\frac{\partial x_\alpha^i}{\partial q_j}\right) = -\frac{\partial V(q_1, \dots, q_n)}{\partial q_j} \qquad V = \sum_{\alpha=1}^{N} V_\alpha$$

Now, I'm going to particularize a problem where all forces depend on a potential. So let's write it down. Let's assume that these forces all derive from a potential. So, I'll just add the pot term to say that we now have derivatives, generalized forces that depend, derive from a potential. The F derives from a potential, that means if I write it in cartesian coordinates that I have to calculate d of v on d of x, alpha i, i is the i-th coordinate, i goes from one to three. And then, here, we have d of xi over d of qj. That's what I wrote here. And now you recognize in this expression something we see often. We have d of v over d of x, d of x over d of qj. So, everything happens as if we had a d of v over d of qj. That's what I noted here with the minus sign, the force is minus the gradient of the potential, where the v, obviously, is the sum over all the potentials. To make things more explicit, I can do the following writing. I say v is a function of x alpha one and x alpha one it's a function of q one up to qn, n number of degree of freedom. Then we have x alpha two, function of q one, qn et cetera. And now I want to calculate d of v over d of q one, for example.

Notes

Summary

25m 44s

Equations de Lagrange (2)

Forces conservatives :

$$Q_j = \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \qquad Q_j^{pot} = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial V_\alpha}{\partial x_\alpha^i}\frac{\partial x_\alpha^i}{\partial q_j}\right) = -\frac{\partial V(q_1,\ldots,q_n)}{\partial q_j} \qquad V = \sum_{\alpha=1}^{N} V_\alpha$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0 \qquad \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot q_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

lagrangien : $L = T - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Avec des forces non-conservatives en plus :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NC}$$

Then q one appears here and there everywhere else so we have to well calculate d of v over d of x alpha one times d of x alpha one over d of q one. That's the kind of term we have here. Now, I can take my Lagrange equation and I'll put a q of this form. You see this term, my minus dv over d of qj, I can put it in here, it goes to the other side of the equal sign, it changes sign so it becomes more, d of v over d of qj. Now, the potential doesn't depend in any way on the q points. It's impossible to have a potential that depends on velocities. So I can add the v in this term for free and I have the equation which has this structure. Then I define what is called the Lagrangian, L which is T minus V and my Lagrangian equation takes that form. You could ask yourself, what happens if we have forces that derive from the potential and forces that don't? So, those that don't derive from a potential, those that derive from a potential, we can do this exercise, go to the other side. And, for the others, we leave them here. And so we have, in all generality, an L which is defined for all the forces which derive from the potential and here the generalized forces, NC for non-conservative, those which do not derive from a potential.

Notes

Summary

27m 50s

Equations de Lagrange (2)

Forces conservatives :

$$Q_j = \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \qquad Q_j^{pot} = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial V_\alpha}{\partial x_\alpha^i}\frac{\partial x_\alpha^i}{\partial q_j}\right) = -\frac{\partial V(q_1,\ldots,q_n)}{\partial q_j} \qquad V = \sum_{\alpha=1}^{N} V_\alpha$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0 \qquad \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

lagrangien : $L = T - V$ $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

Avec des forces non-conservatives en plus : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NC}$

So there you have all the possible forms of Lagrange's equations and now we still have to see how to apply this method to all sorts of mechanical problems. This is what we will do in the next module.

Notes

Summary

29m 46s